Chapter 2: Elements of convex geometry

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Basic definitions. We consider a finite dimensional vector space V over $\mathbb{R}$. Let U be the dual vector space and $\langle \cdot, \cdot \rangle \colon U \times V \longrightarrow \mathbb{R}$ be the natural pairing.

We will refer sometimes to an integral structure on V. By this we mean that we have a finitely generated free abelian group N, and we consider the natural embeddings $N \subset N_{\mathbb{Q}} \subset N_{\mathbb{R}} = V$, where $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ (therefore N is a **lattice** in V). In this case we put $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ so we have $U = M_{\mathbb{R}}$, and $\langle \cdot, \cdot \rangle$ induces the duality between M and N.

We recall the following definitions. A subset $Z \subseteq V$ is called **convex** if for every u_1, u_2 in Z, the whole segment joining u_1 and u_2 is contained in Z, i.e. for every λ in $[0, 1]$, we have $\lambda u_1 + (1 - \lambda)u_2$ in Z. We call Z a **cone** if for every u in Z and every nonnegative real number λ, we have λu in Z. Therefore Z is a convex cone if and only if it is a cone, and for every two vectors in Z, their sum is in Z.

It follows from the definition that an intersection of cones or of convex sets is again a cone or a convex set. Suppose now that $Z \subseteq V$ is an arbitrary subset. The **convex hull** $\mathrm{conv}(Z)$ of Z is the intersection of all convex sets containing Z, i.e. it is the smallest convex set which contains Z. Similarly, the **convex cone generated by** Z is the intersection of all convex cones containing Z, so it is the smallest convex cone which contains Z. We denote it by $\mathrm{pos}(Z)$. If $\sigma = \mathrm{pos}(Z)$, we say that Z is a set of generators of σ.

Lemma 1. *For every subset Z of V, we have*

$$\mathrm{conv}(Z) = \left\{ \sum_{i=1}^{r} \lambda_i v_i \mid \lambda_i \geq 0, \sum_{i=1}^{r} \lambda_i = 1, v_i \in Z \right\}.$$

Proof. The right-hand side is convex and contains Z. Moreover, it is contained in every convex set containing Z, so it is equal to $\mathrm{conv}(Z)$.

We similarly get the following description of the convex cone generated by Z.

Lemma 2. *For every subset Z of V, we have*

$$\mathrm{pos}(Z) = \left\{ \sum_{i=1}^{r} \lambda_i v_i \mid \lambda_i \geq 0, v_i \in Z \right\}.$$

A **convex polytope** (or simply a polytope) is the convex hull of finitely many vectors in V. A **polyhedral** convex cone is the convex cone generated by finitely many vectors. Our main goal in this chapter is to review the basic theory of polyhedral convex cones, and of convex polytopes.

In the presence of the lattice $N \subset V$, we call a convex polytope **rational**, respectively a **lattice polytope** if it is the convex hull of finitely many elements of $N_{\mathbb{Q}}$, respectively in N. Similarly, a convex cone is **rational** if it is generated by elements in $N_{\mathbb{Q}}$ (in this case it can be generated also by elements in N). Whenever we use the adjective "rational", we assume that we are given a lattice $N \subset V$.

We say that a convex cone σ is **pointed** (or **strongly convex**) if whenever both v and $-v$ are in σ, we have $v = 0$. An arbitrary convex cone σ is (noncanonically) the product of a vector space and a pointed convex cone, as follows. Let

$$W = \sigma \cap (-\sigma) := \{v \in \sigma \mid -v \in \sigma\}$$

be the largest vector subspace of V which is contained in σ. If $p\colon V \to V/W$ is the canonical projection, then $p(\sigma)$ is a convex cone of V/W. Moreover, if we choose a splitting i of p, then we get an isomorphism $V \simeq W \times V/W$ that identifies σ with $\simeq W \times p(\sigma)$. Note that by construction, $p(\sigma)$ is pointed. In addition, σ is polyhedral if and only if $p(\sigma)$ is polyhedral.

Exercise. Show that if v_1 and v_2 are in σ and v_1+v_2 is in W, then both v_1 and v_2 are in W. Deduce that if σ is rational, then $N \cap W$ is a lattice in W. Moreover, $p(N)$ is a lattice in V/W and $N \simeq (N\cap W)\times p(N)$; $p(\sigma)$ is rational. Conversely, if W is generated by elements in N and $p(\sigma)$ is rational, then σ is rational.

Lemma 3. *All polytopes and all polyhedral convex cones are closed in V.*

Proof. For every polytope P there are $v_1, \ldots, v_N$ in V such that P is the image of the map

$$\left\{\lambda = (\lambda_i) \in ([0,1])^N \mid \sum_i \lambda_i = 1\right\} \to V,$$

which takes λ to $\sum_i \lambda_i v_i$. Therefore P is compact, hence closed.

Suppose now that σ is a polyhedral convex cone. In order to show that σ is closed in V, we may assume that it is a pointed cone, and that $\sigma \neq \{0\}$. Choose nonzero $v_1, \ldots, v_r$ in V such that σ is the convex cone generated by these vectors. Let P be the convex hull of $v_1, \ldots, v_r$. It follows from Lemmas 1 and 2 that $\sigma = \{\lambda v \mid v \in P, \lambda \geq 0\}$.

Suppose now that $\{\lambda_m v_m\}_m$ converges to w, where $\lambda_m \geq 0$ and $v_m \in P$. By the compactness of P, we may assume after passing to a subsequence that $\{v_m\}_m$ converges to some $v \in P$. Since σ is a pointed cone, 0 is not in P, so $v \neq 0$. Therefore $\{\lambda_m\}_m$ is bounded, and after passing again to a subsequence, we may assume that it converges to some $\lambda \geq 0$. Therefore $w = \lambda v$ is in σ, so σ is closed.

Closed convex subsets. We give now some general results on closed, convex subsets of V. Let K be such a subset. A **supporting hyperplane** of K is an affine hyperplane H in V such that $K \cap H \neq \emptyset$, and such that K is contained in at least one of the half-spaces determined by H. Such a half-space is a **supporting half-space** of K. A **face** of K is the intersection of K with a supporting hyperplane.

Note that by definition, the empty set is never a face, while K is a face if and only if K is contained in a hyperplane. However, it is sometimes convenient to include $\emptyset$ and K as improper faces.

Proposition 1. *A closed convex subset K of V is the intersection of its supporting half-spaces.*

Proof. We fix a scalar product $(\cdot,\cdot)$ on V, which induces a metric d. We have to show that given a point x outside K, there is a supporting hyperplane H such that x is not in the corresponding supporting half-space.

Since K is closed, we can find $x' \in K$ such that

$$d(x, x') = \min_{y \in K} d(x, y). \tag{1}$$

Note that x' is different from x, as x is not in K. We show that the affine hyperplane H passing through x' and perpendicular to the line xx' satisfies the required property.

After a suitable translation, we may assume that $x' = 0$, so $H = \{v \in V \mid (x, v) = 0\}$. Since $(x, x) > 0$, and $x' \in K \cap H$, it is enough to show that for every $y \in K \smallsetminus \{0\}$, we have $(x, y) < 0$. As 0 and y are in K, which is convex, for every $t \in [0, 1]$ we have ty in K. It follows from (1) that

$$(x, x) \leq (x - ty, x - ty)$$

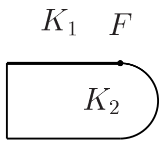
for all $t \in [0, 1]$, and therefore $2(x, y) \leq t(y, y)$ for all $t \in (0, 1]$. This implies $(x, y) \leq 0$, and completes the proof.

We give now some general properties of faces. The **dimension** of a closed, convex set K is the dimension of the affine subspace spanned by K. We denote it by $\dim(K)$. Note that each face of K is again closed and convex, so its dimension is well-defined.

It follows from definition that if F is a face of K, then F is the intersection of K and the affine span of F. Therefore each face of K is determined by its affine span. In particular, if F_1 and F_2 are two faces of K, with F_1 strictly contained in F_2, then $\dim(F_1) < \dim(F_2)$.

A **vertex** of K is a zero-dimensional face of K, and a **facet** is a face F with $\dim(F) = \dim(K) - 1$.

Suppose that we have $F \subseteq K_1 \subseteq K_2$, with K_1 and K_2 closed convex sets, such that F is a face of K_2. Then F is a face of K_1. On the other hand, it is not true in general that if F is a face of K_1, and K_1 is a face of K_2, the F is a face of K_2. Take for example



However, we will see that this is true for polyhedral convex cones and for polytopes.

Lemma 4. *The intersection of finitely many faces of K is again a face of K (possibly empty).*

Proof. Let $F = \bigcap_i F_i$, where each F_i is a face of K. We may assume that F is nonempty. Let u_i be nonzero elements of U, and $a_i \in \mathbb{R}$ be such that for all i we have $F_i = K \cap \{v \mid \langle u_i, v\rangle = a_i\}$, and $\langle u_i, v\rangle \geq a_i$ for every $v \in K$.

Let $u = \sum_i u_i$. If $u = 0$, then for a fixed i we may replace u_i and a_i by $2u_i$ and $2a_i$, respectively, so we may assume $u \neq 0$. As $\langle u, v\rangle \geq \sum_i a_i$ for all $v \in K$, with equality if and only if $v \in \bigcap_i F_i$, we see that F is a face.

Polyhedral convex cones. Let σ be a closed convex cone in V. A supporting half-space of σ is defined by

$$\{v \in V \mid \langle u, v\rangle \geq a\}$$

for some nonzero u in U, and some $a \in \mathbb{R}$. By definition there is v in σ such that $\langle u, v\rangle = a$. Moreover, for every $t \geq 0$ we have tv in σ, so $ta \geq a$. Therefore $a = 0$, hence every supporting hyperplane is a linear hyperplane. We make the convention that σ is considered as a face, but the empty face is not.

The **dual cone** of σ is

$$\sigma^{\vee} := \{u \in U \mid \langle u, v\rangle \geq 0 \,\text{for all}\, v \in \sigma\}.$$

It is clear that $\sigma^\vee$ is a closed convex cone in U. For a subset A of U, we denote by $A^\perp$ the set of all v in V with $\langle u,v\rangle = 0$ for all $u \in A$. A face of σ is therefore a subset of the form $\sigma \cap u^\perp$, for some $u \in \sigma^\vee$.

Proposition 1 implies in this setting

Proposition 2. *If σ is a closed convex cone, then $(\sigma^\vee)^\vee = \sigma$.*

Exercise. If τ is a face of σ, then for every v_1 and v_2 in σ such that $v_1 + v_2$ is in τ, both v_1 and v_2 are in τ.

From now on we assume that σ is polyhedral. Let $v_1, \ldots, v_r$ be generators of σ, and let τ be a face of σ. It follows from the above exercise that τ is the convex cone generated by those v_i which lie in τ, so in particular it is a polyhedral convex cone. If σ is rational, then so are its faces. Moreover, we see that there are only finitely many faces of σ.

Lemma 5. *If τ is a face of σ, and if τ_1 is a face of τ, then τ_1 is a face of σ.*

Proof. Write $\tau = \sigma \cap u^\perp$ and $\tau_1 = \tau \cap u_1^\perp$, with u in $\sigma^\vee$ and u_1 in $\tau^\vee$. Choose generators $v_1, \ldots, v_r$ of σ, indexed such that the first s of them are those that lie in τ.

Since $\langle u, v_i\rangle > 0$ for $i > s$, we see that if $t \gg 0$, we have $\langle u_1 + tu, v_i\rangle > 0$ for such i. Therefore $u_1 + tu \in \sigma^\vee$ and

$$\sigma \cap (u_1 + tu)^\perp = \tau \cap u_1^\perp = \tau_1.$$

Exercise. Let τ be a face of σ, and let W be the linear span of τ. If $p\colon V \to V/W$ is the canonical projection, then the map $\tau' \to p(\tau')$ gives an inclusion-preserving bijection between the faces of σ that contain τ, and the faces of $p(\sigma)$. Its inverse is given by $\tau'' \to \sigma \cap p^{-1}(\tau'')$.

In order to study the faces of σ, we may replace V by the linear span W of σ, as this does not affect the faces. Note also that if we choose a splitting of the inclusion $W \hookrightarrow V$, then $\sigma^\vee$ is isomorphic to the product of the dual of σ in W^* and $(V/W)^*$.

Proposition 3. *Every face τ of σ is the intersection of the facets containing it. If $\tau_1 \subset \tau_2$ are faces of σ, with no other face in between, then* $\dim(\tau_2) = \dim(\tau_1) + 1$. *In addition, if* $\dim(\tau) = \dim(\sigma) - 2$, *then there are precisely two facets containing τ.*

Proof. We may assume that V is the linear span of σ. Moreover, using the above exercise we reduce ourselves to the case $\tau = \{0\}$. Suppose that $\dim(\sigma) \geq 2$, and let $v_1, \ldots, v_r$ be nonzero generators of σ.

We know that there is $u \in \sigma^\vee$ such that $\sigma \cap u^\perp = \{0\}$. On the other hand, as $\dim(V) \geq 2$, we may choose $w \in U \smallsetminus \mathbb{R}u$ such that $\langle w, v_1 \rangle < 0$. If

$$t = \max\{s > 0 \mid u + sw \in \sigma^\vee\},$$

then there is i such that $v_i \in (u + tw)^\perp$. Therefore $\sigma \cap (u + tw)^\perp$ is a proper nonzero face of σ.

This shows that every proper face τ of codimension at least two is strictly contained in a proper face. By induction on the codimension of τ, this implies that every proper face is contained in a facet. Taking τ_2 as the ambient cone gives the second assertion of the proposition.

Moreover, with the previous notation, if $\dim(\sigma) = 2$ and $\tau = \{0\}$, we see that σ has two generators v_1 and v_2, and σ has only two facets, generated by v_1 and v_2. Their intersection is $\{0\}$. By induction on $\dim(\sigma)$ we deduce the first assertion.

Lemma 6. *If the linear span of σ is V, then the topological boundary $\partial\sigma$ of σ in V is the union of its facets. Moreover, σ is the closure of its interior.*

Proof. If τ is a proper face of σ, then τ is the intersection of σ with a supporting hyperplane H, so $\tau \subseteq H \subseteq \overline{V \smallsetminus \sigma}$. This implies that the union of the facets (equivalently, of the proper faces) is contained in the boundary of σ. Moreover, if x and y are in σ, and if y does not lie on any proper face, then the same property holds for every z in the open segment joining x and y.

In order to finish, it is enough to show that if v is in σ, but in none of the facets, then it lies in the topological interior of σ. Indeed, suppose that we have a sequence of vectors v_m not in σ such that $\lim_{m\to\infty} v_m = v$. By Proposition 2, we can find $u_m \in \sigma^\vee$ such that $\langle u_m, v_m \rangle < 0$. We may assume that all u_m lie on a sphere around the origin. Because the sphere is compact, by passing to a subsequence, we may assume $u_m \to u$ for some nonzero u. Note that u is in $\sigma^\vee$ (as $\sigma^\vee$ is closed) and $\langle u, v \rangle \leq 0$. It follows that v lies in the proper face $\tau = \sigma \cap u^\perp$, a contradiction.

When σ does not necessarily span V, the complement in σ of the union of all the facets is called the **relative interior** of σ. By Lemma 6, this is the topological interior of σ in the linear span of σ. In order to write down an element v in the relative interior of σ, one can take generators $v_1, \ldots, v_r$, and then $v = \sum_i m_i v_i$ where all m_i are positive.

Keeping the assumption that V is the linear span of σ, we see that every facet τ of σ is given by $\tau = \sigma \cap u_\tau^\perp$ for some u_τ in $\sigma^\vee$, uniquely determined up to multiplication by a positive real number.

Lemma 7. *With the above notation, we have*

$$\sigma = \{v \in V \mid \langle u_\tau, v\rangle \geq 0 \, \textit{for all facets} \, \tau\}.$$

Proof. Suppose that for some v, we have $\langle u_\tau, v\rangle \geq 0$ for all facets τ. If v is not in σ, choose w in the interior of σ. If

$$t = \min\{s > 0 \mid v + sw \in \sigma\},$$

then there is a facet τ such that $v + tw \in \tau$. Since $\langle u_\tau, v + tw\rangle = 0$, $\langle u_\tau, v\rangle \geq 0$, and $\langle u_\tau, w\rangle > 0$ (w is an interior point), we get a contradiction.

We deduce that the u_τ generate $\sigma^\vee$. Indeed, if σ_1 is the convex cone generated by the u_τ, then it follows from Lemma 7 that $\sigma = \sigma_1^\vee$. On the other hand, as σ_1 is polyhedral, we have $\sigma_1 = (\sigma_1^\vee)^\vee = \sigma^\vee$ by Proposition 2.

Therefore we have

Proposition 4 (Farkas)**.** *If σ is a polyhedral convex cone, then so is its dual $\sigma^\vee$.*

Proof. We may assume that V is the linear span of σ, and in this case we have given finitely many generators of $\sigma^\vee$.

It follows from this proposition and Lemma 7 that a subset of V is the convex cone generated by finitely many vectors if and only if it is the intersection of finitely many linear half-spaces.

The following is a practical algorithm for finding at the same time the facets of σ, as well as generators for $\sigma^\vee$. Suppose that σ is generated by $v_1, \ldots, v_r$, which also generate V as a vector space. If $n = \dim(V)$, for every set $v_{i_1}, \ldots, v_{i_{n-1}}$ of linearly independent vectors, let $u \in U$ be such that $\langle u, v_{i_\alpha}\rangle = 0$ for all α. If $\langle u, v_j\rangle \geq 0 (\leq 0)$ for all j, then $\sigma \cap u^\perp$ is a facet of σ, and u (respectively, $-u$) is taken as one of the generators of $\sigma^\vee$. If this is not the case, then u is discarded. The vectors we obtain from all possible subsets of the v_j, as above, are the generators of $\sigma^\vee$, and correspondingly we get all the facets of σ. Of course, different choices of subsets may give rise to the same facet, hence to proportional generators.

By repeating the above procedure one can give a description of all the faces of σ starting from its generators.

Remark 1. Suppose that σ is rational. If V is the linear span of σ, then for every facet τ, we may take $u_\tau \in M$. It follows that in general, $\sigma^\vee$ is rational. Moreover, we see that if v is not in σ, then there is u in $\sigma^\vee \cap M$ such that $\langle u, v\rangle < 0$. Indeed, we may suppose that σ spans V, and then we may take a suitable u_τ by Lemma 7.

Exercise. Show that if σ_1 and σ_2 are polyhedral convex cones, then so are $\sigma_1 + \sigma_2 := \{v_1 + v_2 \mid v_1 \in \sigma_1,\, v_2 \in \sigma_2\}$ and $\sigma_1 \cap \sigma_2$. Moreover, we have $(\sigma_1 + \sigma_2)^\vee = \sigma_1^\vee \cap \sigma_2^\vee$ and $(\sigma_1 \cap \sigma_2)^\vee = \sigma_1^\vee + \sigma_2^\vee$.

We describe now the correspondence between the faces of σ and those of $\sigma^\vee$.

Proposition 5. *For every face τ of a polyhedral convex cone σ, $\tau^* := \sigma^\vee \cap \tau^\perp$ is a face of $\sigma^\vee$. This gives an inclusion-reversing bijection between the faces of σ and the faces of $\sigma^\vee$, such that $(\tau^*)^* = \tau$ and*

$$\dim(\tau) + \dim(\tau^*) = \dim(V). \tag{2}$$

Proof. If v is a point in the relative interior of a face τ of σ, then $\sigma^\vee \cap \tau^\perp = \sigma^\vee \cap v^\perp$: otherwise we could find $u \in \sigma^\vee$ such that $\tau \cap u^\perp$ is a proper face of τ containing v. This shows that τ^* is a face of $\sigma^\vee$.

Moreover, every face of $\sigma^\vee$ arises in this way: if $v \in \sigma$ gives the face $\sigma^\vee \cap v^\perp$, then it is enough to pick τ the smallest face of σ containing v. It is clear that the map taking τ to τ^* reverses inclusions, and that $\tau \subseteq (\tau^*)^*$. This implies $\tau^* = ((\tau^*)^*)^*$, and as we have already seen the surjectivity of the map, we deduce that $\tau = (\tau^*)^*$ for every τ.

Since $\sigma^* = \sigma^\perp$, we see that (2) holds if $\tau = \sigma$. For a general τ it is enough to take a sequence of faces of σ between τ and σ, each having codimension one in the larger one: from what we have already proved, the corresponding sequence of faces of $\sigma^\vee$ has the same property.

Remark 2. Note that the linear span of τ^* is $\tau^\perp$. Indeed, by definition the linear span of τ^* is contained in $\tau^\perp$, and we must have equality since both have the same dimension.

Remark 3. If σ is a rational cone, then we can find in the relative interior of σ a point in N: if $v_1, \ldots, v_r$ are generators of σ in N, then any linear combination of the v_i with positive integer coefficients has this property. If τ is any face of σ, we see that we can find $u \in M \cap \sigma^\vee$ such that $\tau = \sigma \cap u^\perp$. Indeed, take any u in M which is in the relative interior of τ^*.

The smallest face of σ is $(\sigma^\vee)^*$, hence it is equal to $\sigma \cap (-\sigma)$. We see that σ is a pointed cone if and only if $\{0\}$ is a face. A **ray** of a pointed cone is a one-dimensional face.

Remark 4. If σ is a pointed polyhedral convex cone, and if $v_1, \ldots, v_r$ is a minimal set of generators of σ, then $\mathbb{R}_+ v_1, \ldots, \mathbb{R}_+ v_r$ are the rays of σ. Therefore such a minimal set of generators is unique up to reordering and multiplication by positive scalars. Indeed, if τ is a ray of σ, then it is generated by the v_i in τ, so there is i such that $\tau = \mathbb{R}_+ v_i$. Conversely, given $i \leq r$, as $v_i \notin \mathrm{pos}(v_j \mid j \neq i)$ we can find $u \in U$ such that

$\langle u, v_i \rangle < 0$ and $\langle u, v_i \rangle \geq 0$ for all $j \neq i$. On the other hand, as σ is a pointed cone, there is w in U such that $\langle w, v_j \rangle > 0$ for all j. If $t = -\langle u, v_i \rangle / \langle w, v_i \rangle$, then $u + tw \in \sigma^\vee$ and $\mathbb{R}_+ v_i = \sigma \cap (u + tw)^\perp$.

Proposition 6. *If τ is a face of σ, and if $u \in \sigma^\vee$ is such that $\tau = \sigma \cap u^\perp$, then $\tau^\vee = \sigma^\vee + \mathbb{R}_+(-u)$.*

Proof. It is enough to show that given u' in $\tau^\vee$, if $t \gg 0$, then $u' + tu$ is in $\sigma^\vee$. This follows as in the proof of Lemma 5.

Proposition 7. *Let σ_1 and σ_2 be polyhedral convex cones in V. If $\sigma_1 \cap \sigma_2 = \tau$ is a face of both σ_1 and σ_2, then there is u in $\sigma_1^\vee \cap (-\sigma_2)^\vee$ such that $\tau = \sigma_1 \cap u^\perp = \sigma_2 \cap u^\perp$. Moreover, if σ_1 and σ_2 are rational, then we may take u in M.*

Proof. Consider the polyhedral convex cone

$$\gamma = \sigma_1 - \sigma_2 := \{v_1 - v_2 \mid v_1 \in \sigma_1, v_2 \in \sigma_2\},$$

and take u in the relative interior of $\gamma^\vee$ (we may choose $u \in M$ if both σ_1 and σ_2 are rational). Note that

$$\gamma \cap u^\perp = \gamma \cap (-\gamma) = (\sigma_1 - \sigma_2) \cap (\sigma_2 - \sigma_1). \tag{3}$$

As $u \in (\sigma_1 - \sigma_2)^\vee$, we get that $u \in \sigma_1^\vee \cap (-\sigma_2)^\vee$. In particular, we have $\tau \subseteq u^\perp$.

Suppose now that $v \in \sigma_2 \cap u^\perp$. By (3) we can write $v = v_1 - v_2$ with $v_1 \in \sigma_1$ and $v_2 \in \sigma_2$. We have $v + v_2 = v_1 \in \tau$, so $v \in \tau$ as τ is a face of σ_2. Therefore $\sigma_2 \cap u^\perp = \tau$, and similarly we get $\sigma_1 \cap u^\perp = \tau$.

Note that in the above proof we have only used the fact that if v and v_2 are in σ_2, and if $v + v_2$ is in τ, then v and v_2 are in τ (and similarly for σ_1). By taking an arbitrary convex cone $\tau \subseteq \sigma$ with this property, and applying the above argument for $\sigma_1 = \tau$ and $\sigma_2 = \sigma$ we get the following

Proposition 8. *If σ is a polyhedral convex cone, and if τ is a subset of σ, then τ is a face of σ if and only if τ is a convex cone such that for every v_1 and v_2 in σ with $v_1 + v_2$ in τ, we have v_1 in τ and v_2 in τ. Moreover, if σ is rational, then it is enough for this condition to hold for v_1 and v_2 in N.*

We include the following lemma for future reference.

Lemma 8 (Carathéodory)**.** *Let σ be the convex cone generated by a set T. Then σ is the union of the convex cones generated by subsets of T consisting of linearly independent vectors.*

Proof. It is enough to show that if $v = \lambda_1 v_1 + \ldots \lambda_r v_r$, with $\lambda_i > 0$ for all i, and if $v_1, \ldots, v_r$ are not linearly independent, then v can be written as a linear combination with positive coefficients of $r-1$ of the v_i. For this, consider a relation $a_1 v_1 + \ldots + a_r v_r = 0$, where some of the a_i are nonzero. After possibly multiplying the relation by (-1), we may assume that there is j such that $a_j > 0$.

Let i be such that $a_i > 0$ and $\lambda_i / a_i = \min\{\lambda_j / a_j \mid a_j > 0\}$. Note that we have $\lambda_j - \lambda_i \frac{a_j}{a_i} \geq 0$ for all j. Therefore we can write

$$v = \sum_{j \neq i} \left(\lambda_j - \lambda_i \frac{a_j}{a_i} \right) v_j,$$

and all coefficients are non-negative.

Cones and semigroups. Let σ be a rational polyhedral convex cone in $V = N_{\mathbb{R}}$, and let $S := \sigma \cap N$.

Proposition 9 (Gordan)**.** *S is a finitely generated semigroup.*

Proof. Let $v_1, \ldots, v_r \in N$ be generators of σ. The set

$$K := \left\{ \sum_{i=1}^{r} \lambda_i v_i \mid \lambda_i \in [0,1] \quad \text{for all} \quad i \right\}$$

is compact (note that it contains also the v_i). As N is discrete in V, its intersection with K is finite: let $w_1, \ldots, w_s$ be its elements. If $v \in S$, and if we write $v = \sum_{i=1}^{r} \alpha_i v_i$, with $\alpha_i \geq 0$, then there is j such that $v = \sum_{i=1}^{r} \lfloor \alpha_i \rfloor v_i + w_j$, so S is generated as a semigroup by $w_1, \ldots, w_s$.

Lemma 9. *If V is the linear span of σ, then S^{gp} is equal to N.*

Proof. Let $e_1, \ldots, e_n$ be a basis of N. We choose $v \in N$ in the interior of σ. By Lemma 6, σ contains a ball around v, so if m is a large enough integer, then each $mv + e_i$ is in σ. Hence $e_i \in S^{\mathrm{gp}}$ for all i.

Proposition 10. *Let $\mathbb{R}_+ S$ denote the convex cone generated by the semigroup S. The assignment $S \longrightarrow \mathbb{R}_+ S$ gives an inclusion-preserving bijection between finitely generated sub-semigroups of N which are saturated in N, and rational polyhedral convex cones σ in V. The inverse of this map is given by $\sigma \longrightarrow \sigma \cap N$. Moreover, S^{gp} is equal to N if and only if σ spans V.*

Proof. It follows from Lemma 9 and Proposition 9 that our maps are well-defined, and $\sigma = \mathbb{R}_+(\sigma \cap N)$ by the definition of rational cones. Therefore it is enough to show that if S is a semigroup as in the proposition, then $S = \mathbb{R}_+ S \cap N$. If $v \in \mathbb{R}_+ S \cap N$, then by Lemma 8 we can find $v_1, \ldots, v_r \in S$, linearly independent over $\mathbb{R}$, such that $v = \sum_i \lambda_i v_i$, with all $\lambda_i \geq 0$. Since the v_i can be completed to a basis of $N_{\mathbb{Q}}$ and v is in $N_{\mathbb{Q}}$, it follows that all λ_i are in $\mathbb{Q}$. Therefore there is a positive integer m with mv in S. As v is in N and S is saturated in N, we see that v is in S.

Remark 5. Let σ be a rational polyhedral convex cone in V, and let V' be the linear span of σ. Then $N' := V' \cap N$ is a subgroup of N such that N/N' is torsion-free. The canonical embedding $N'_{\mathbb{R}} \hookrightarrow V'$ is an isomorphism, since V' is generated by $V' \cap N$. We can use this to reduce many questions to the case when V is the linear span of σ. For example, Lemma 9 implies that the dimension of $S = \sigma \cap N$ (which by definition is the rank of S^{gp}) is equal to $\dim(\sigma)$. Similarly, the same lemma shows that S^{gp} is equal to the intersection between N and the linear span of σ.

Corollary 1. *The bijection in Proposition 10 induces a bijection between the faces of σ and those of $S = \sigma \cap N$.*

Proof. It is enough to show that if T is a face of S, then $\mathbb{R}_+T$ is a face of $\sigma = \mathbb{R}_+S$. As $\mathbb{R}_+S \cap N = S$ and $\mathbb{R}_+T \cap N = T$, this follows from the last assertion in Proposition 8.

If S is a sub-semigroup of N, let

$$S^\vee := \{u \in M \mid \langle u, v\rangle \geq 0 \quad \text{for all} \quad v \in S\}.$$

If S is finitely generated and saturated in N, and if $\sigma = \mathbb{R}_+S$, then $S^\vee = \sigma^\vee \cap M$, so $S^\vee$ is again finitely generated and saturated in M, and $(S^\vee)^\vee = S$.

Note that if S is a finitely generated semigroup with $S^{\mathrm{gp}} = N$, then $\mathbb{R}_+S \cap N$ is equal to S^{sat}.

Remark 6. If $S^\vee = \sigma^\vee \cap M$, then a face of $S^\vee$ is of the form $T = \sigma^\vee \cap \tau^\perp \cap M$. In this case we have $T^{\mathrm{gp}} = \tau^\perp \cap M$. Indeed, T^{gp} is equal to the intersection of M with the linear span of $\sigma^\vee \cap \tau^\perp$, which is equal to $M \cap \tau^\perp$ by Remark 2.

Remark 7. Let σ be a rational polyhedral convex cone and let τ be a face of σ. We choose $u \in \sigma^\vee \cap M$ such that $\tau = \sigma \cap u^\perp$. If $S = \sigma \cap N$ and $T = \tau \cap N$, then $T^\vee = S^\vee + \mathbb{N}(-u)$. Indeed, we have seen in the proof of Lemma 5 that if $u' \in \tau^\vee$, then $u' + mu \in \sigma^\vee$ for m large enough.

Remark 8. Let σ_1 and σ_2 be rational polyhedral convex cones such that the intersection τ of σ_1 and σ_2 is a face of both σ_1 and σ_2. If S_1, S_2 and T are the corresponding semigroups, then $T^\vee = S_1^\vee + S_2^\vee$. Indeed, it follows from Proposition 7 that there is u in $M \cap \sigma_1^\vee \cap (-\sigma_2)^\vee$ such that $T = S_1 \cap u^\perp = S_2 \cap u^\perp$. It follows from the previous remark that

$$T^\vee = S_1^\vee + \mathbb{N}(-u) \subseteq S_1^\vee + S_2^\vee,$$

and the reverse inclusion follows from the fact that T is contained in S_1 and S_2.

Let σ be a rational polyhedral convex cone and let $S = \sigma \cap N$ be the corresponding semigroup. Note that S has no nonzero invertible elements if and only if σ is a pointed cone.

Suppose that S is a finitely generated sub-semigroup of N, not necessarily saturated, such that S has no nonzero invertible elements. A nonzero v in S is called **irreducible** if whenever $v = v_1 + v_2$ with v_1 and v_2 in S, we have $v_1 = 0$ or $v_2 = 0$.

Exercise. Show that every nonzero element in S can be written as a sum of finitely many irreducible elements. Deduce that there is a unique minimal set of generators of S, which consists of all irreducible elements in S. This is called the **Hilbert basis** of S.

Remark 9. One of the basic open problems on semigroups is to characterize saturation in terms of the combinatorial properties of the Hilbert basis. We refer to [BGT] for a survey on this and related questions.

We want to stress that many questions related to semigroups are much more delicate than the corresponding questions for cones. For example, one says that a semigroup S as in the above exercise satisfies the **integral Carathéodory property** (compare with Lemma 8) if every element v in S can be represented as $v = m_1 v_1 + \ldots + m_r v_r$ with $m_i \in \mathbb{N}$, v_i in the Hilbert basis of S and $r \leq \dim(S)$. It is known that this holds whenever $\dim(S) \leq 3$ (see [Se]), but there is an example in dimension 6 of a saturated semigroup which does not satisfy this property (see [BGHMW]).

Polytopes and polyhedra. A **convex polyhedron** (or simply a polyhedron) is the intersection of finitely many affine half-spaces. It follows from Proposition 4 that a polyhedral cone is a convex polyhedron, and we will see in Corollary 2 below that the same is true for a polytope.

In the presence of a lattice $N \subset N_{\mathbb{R}} = V$, we say that a polyhedron is **rational** if the half-spaces are of the form $\{v \mid \langle u, v \rangle \geq a\}$ for some $u \in M_{\mathbb{Q}}$ and $a \in \mathbb{Q}$. Note that if this is the case, we may take in fact $u \in M$ and $a \in \mathbb{Z}$.

Note that the face of a polyhedron is again a polyhedron. Let K be a nonempty polyhedron, and let $u_1, \ldots, u_r$ in $U \setminus \{0\}$, $a_1, \ldots, a_r$ in $\mathbb{R}$

be such that

$$K = \{v \in V \mid \langle u_i, v\rangle \geq a_i \quad \text{for all} \quad i\}.$$

Consider the closed cone σ over the subset $K \times \{1\}$ of $V \times \mathbb{R}$, i.e.

$$\sigma := \overline{\{(v,t) \in V \times \mathbb{R} \mid t > 0, (1/t)v \in K\}}.$$

Note that $\sigma = \{(v,t) \mid t \geq 0, \langle u_i, v\rangle \geq a_i t \, \text{for all}\, i\}$. Indeed, if we denote the right-hand side by σ', then σ' is a polyhedral convex cone by Proposition 4, and its intersection with $V \times (\mathbb{R}^* \setminus \{0\})$ is equal to $\{(v,t) \mid t > 0, (1/t)v \in K\}$. Moreover, $\sigma' \cap (V \times \{0\})$ is a proper face of σ', so σ' is the closure of $\sigma' \smallsetminus (V \times \{0\})$.

Lemma 10. *With the above notation, the following are equivalent:*

(1) *K is bounded.*
(2) *K is compact.*
(3) $\sigma \cap (V \times \{0\}) = \{(0,0)\}$.
(4) *K is a convex polytope.*

Proof. As K is closed, (1) and (2) are equivalent. Suppose now that (2) holds. If $(v,0)$ lies in σ, then there are $t_m > 0$ and $v_m \in K$ such that $\lim_{m\to\infty}(t_m v_m, t_m) = (v,0)$. As $\{v_m\}_m$ is bounded and $\lim_{m\to\infty} t_m = 0$, we deduce $v = 0$, so (2) implies (3).

We show now (3)$\Rightarrow$(4). We have seen that σ is a polyhedral convex cone, and by (3) we may choose generators of σ of the form $\{(v_1,1),\ldots,(v_s,1)\}$. Using the descriptions in Lemmas 1 and 2, we see that $K = \mathrm{conv}(v_1,\ldots,v_s)$. As we know that every polytope is compact, this completes the proof.

Corollary 2. *A subset K of V is a (rational) polytope if and only if it is a bounded (rational) polyhedron.*

Proof. We have already seen that a bounded polyhedron K is a polytope. With the notation in the lemma, we see that if K is rational, then σ is rational, and hence we may choose the generators such that $v_i \in N_{\mathbb{Q}}$ for all i. Therefore K is a rational polytope.

Conversely, suppose that P is a polytope, and let σ be the cone generated by the subset $P \times \{1\}$ of $V \times \mathbb{R}$, so $\sigma = \{0\} \cup \{(v,t) \mid t > 0, (1/t)v \in P\}$. In particular, we have $P \times \{1\} = \sigma \cap (V \times \{1\})$.

If $P = \mathrm{conv}(v_1,\ldots,v_s)$, then σ is generated by the $(v_i,1)$, so it is polyhedral. If we choose $u_1,\ldots,u_r$ in U, and $a_1,\ldots a_r$ in $\mathbb{R}$ such that

$$\sigma = \{(v,t) \mid \langle u_i, v\rangle \geq a_i t \quad \text{for all} \quad i\},$$

then $P = \{v \mid \langle u, v_i\rangle \geq a_i \, \text{for all}\, i\}$. Note that if P is rational as a polytope, then σ is rational, so we may choose $u_i \in M$ and $a_i \in \mathbb{Z}$.

Exercise. Show that every pointed cone can be written as the cone over a polytope in a suitable affine hyperplane of V, not containing the origin.

The connection between polyhedra and cones that we have used in Proposition 10 can be further exploited to deduce results about the faces of polyhedra from what we already know about faces of cones. In what follows, for a polyhedron P we consider P as an improper face, but not the empty set.

Lemma 11. *Let P be a convex polyhedron. If σ is the closed cone over $P \times \{1\}$, then we have $\dim(\sigma) = \dim(P) + 1$. If τ is a face of σ not contained in the face $\sigma_0 := \sigma \cap \{(v,t) \mid t = 0\}$, then $\tau \cap (V \times \{1\}) = Q \times \{1\}$ for a face Q of P such that τ is the closed cone over $Q \times \{1\}$. This gives an inclusion-preserving bijection between the faces of σ not contained in σ_0 and the faces of P.*

Proof. A face of P is of the form $Q = P \cap \{v \mid \langle u, v\rangle = b\}$ for some u in M such that $\langle u, v\rangle \geq b$ for all v in P. This implies that for every (v,t) in σ, we have $\langle u, v\rangle \geq bt$. The closed cone τ over Q is equal to $\sigma \cap \{(v,t) \mid \langle u, v\rangle = bt\}$, so it is a face of σ and we have $\tau \cap V \times \{1\} = Q \times \{1\}$. Since Q is nonempty, it is clear that τ is not contained in σ_0.

Conversely, let τ be a face of σ not contained in σ_0. We write $\tau = \sigma \cap \{(v,t) \mid \langle u, v\rangle + st = 0\}$ where (u,s) is such that for all (v,t) in σ we have $\langle u, v\rangle + st \geq 0$.

Since τ is not contained in σ_0, if we put $\tau \cap (V \times \{1\}) = Q \times \{1\}$, we see that Q is nonempty. We have $Q = P \cap \{v \mid \langle u, v\rangle + s = 0\}$, so Q is a face of P such that τ is the closed cone over $Q \times \{1\}$.

We can deduce the basic properties of the faces of a polyhedron P from the above lemma and the corresponding properties of faces of polyhedral cones. For example, we see that P has only finitely many faces and that the face of a face of P is again a face of P. Moreover, every face of P is an intersection of facets. If P is rational, then all the faces of P are rational. We also see that if $\dim(P)$ is equal to $\dim(V)$, then the topological boundary of P is the union of all its facets.

Remark 10. Suppose now that P is a polytope. We have seen that given $v_1, \ldots, v_r$ in V, we have $P = \mathrm{conv}(v_1, \ldots, v_r)$ if and only if $(v_1, 1), \ldots, (v_r, 1)$ generate σ. As σ is a pointed cone by Lemma 10, its minimal generators correspond to its one-dimensional faces. This shows that there is a unique minimal set Γ in V such that $P = \mathrm{conv}(\Gamma)$, namely the set of all vertices of P.

Minkowski sums. Let P and Q be convex subsets of V. The **Minkowski sum** of P and Q is the set

$$P + Q := \{v_1 + v_2 \mid v_1 \in P, v_2 \in Q\}.$$

Note that $P + Q$ is convex.

Example 1. If P is convex, then the mth iterated Minkowski sum $P + \ldots + P$ is equal to mP.

Example 2. If σ_1 and σ_2 are (polyhedral) convex cones, then so is $\sigma_1 + \sigma_2$. A generating set for $\sigma_1 + \sigma_2$ is the union of generating sets for σ_1 and σ_2.

Example 3. If P and Q are convex polytopes, then so is $P + Q$. Moreover, if P is the convex hull of $v_1, \ldots, v_r$ and Q is the convex hull of $w_1, \ldots, w_s$, then $P + Q$ is the convex hull of the $v_i + w_j$ for $i \leq r$, $j \leq s$. Indeed, the convex hull of these elements is contained in $P + Q$. On the other hand, a vector in $P + Q$ can be written as $v = \sum_{i=1}^{r} \lambda_i v_i + \sum_{j=1}^{s} \mu_j w_j$, where λ_i and μ_j are in $\mathbb{R}_+$, with $\sum_i \lambda_i = 1$ and $\sum_j \mu_j = 1$. In this case there are $c_{i,j}$ for $1 \leq i \leq r$ and $1 \leq j \leq s$ such that $\sum_j c_{i,j} = \lambda_i$ for all i, and $\sum_i c_{i,j} = \mu_j$ for all j. (Exercise: prove this!) It follows that $\sum_{i,j} c_{i,j} = 1$, and $v = \sum_{i,j} c_{i,j}(v_i + w_j)$.

Consider now a convex polyhedron P, and let σ be the closed cone over $P \times \{1\}$. We have seen that σ is a polyhedral convex cone, so we can choose generators $(v_1, 1), \ldots, (v_r, 1)$, $(w_1, 0), \ldots, (w_s, 0)$. Note that if σ_0 is the convex cone generated by $w_1, \ldots, w_s$, then $\sigma_0 \times \{0\}$ is the face $\sigma \cap \{(v, t) \mid t = 0\}$ of σ. Let Q be the convex hull of $v_1, \ldots, v_r$. Since $P \times \{1\} = \sigma \cap (V \times \{1\})$, we see that $P = \sigma_0 + Q$.

Conversely, suppose that Q is a polytope and σ_0 is a polyhedral convex cone, and let $P = Q + \sigma_0$. If we put $Q = \mathrm{conv}(v_1, \ldots, v_r)$ and $\sigma_0 = \mathrm{pos}(w_1, \ldots, w_s)$, then

$$\sigma := \mathrm{pos}((v_1, 1), \ldots, (v_r, 1), (w_1, 0), \ldots, (w_s, 0))$$

is a polyhedral convex cone such that $P \times \{1\} = \sigma \cap (V \times \{1\})$. Therefore P is a convex polyhedron.

Note that σ_0 is uniquely determined by P. In fact, if P is given by $\langle u_\ell, v \rangle \geq a_\ell$ for ℓ in a set I, then σ_0 is the dual of the cone generated by the u_ℓ. On the other hand, while the polytope Q is not unique, when σ_0 is a pointed cone we have a canonical choice for Q, namely the convex hull of the vertices of P. Indeed, in this case the cone σ over $P \times \{1\}$ is a pointed cone, and $(v, 1)$ is a minimal generator of σ if and only if v is a vertex of P. In particular, we have proved the following

Proposition 11. *A subset P of V is a convex polyhedron if and only if it is the Minkowski sum of a polytope and a polyhedral convex cone.*

Remark 11. With the above notation, P is a rational polyhedron if and only if the closed cone σ over $P \times \{1\}$ is rational. We see that if $P = Q + \sigma_0$ and both Q and σ_0 are rational, then so is P. Conversely, if P is rational and if we take σ_0 and Q as in Proposition 11, they are both rational. Similar considerations

Remark 12. We have seen that the Minkowski sum of two polytopes or polyhedral convex cones is again a polytope or a polyhedral convex cone. It therefore follows from Proposition 11 that the Minkowski sum of two convex polyhedra is again a convex polyhedron.

Let σ be a rational polyhedral convex cone, and let $S = \sigma \cap N$. We have seen in Chapter 1 that the ideal defining a closed subscheme of $\mathrm{Spec}(k[S])$ which is invariant under the torus action is **monomial**, i.e. it is generated by elements of the form χ^v for v in S. If $\mathfrak{a}$ is a nonzero monomial ideal in $k[S]$, the **Newton polyhedron** $P_{\mathfrak{a}}$ of $\mathfrak{a}$ is the convex hull of $\{v \in S \mid \chi^v \in \mathfrak{a}\}$.

Exercise. Show that if $\chi^{v_1}, \ldots, \chi^{v_n}$ are generators of $\mathfrak{a}$ and if Q is the convex hull of $v_1, \ldots, v_r$, then $P_{\mathfrak{a}} = Q + \sigma$, so $P_{\mathfrak{a}}$ is a rational convex polyhedron. Show that if $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals, then

$$P_{\mathfrak{a}\mathfrak{b}} = P_{\mathfrak{a}} + P_{\mathfrak{b}}.$$

Exercise. With the notation in the previous exercise, show that the ideal $(\chi^v \mid v \in P_{\mathfrak{a}})$ is the **integral closure** $\overline{\mathfrak{a}}$ of $\mathfrak{a}$: this is the set of all $f \in k[S]$ which satisfy an equation of the form

$$f^m + c_1 f^{m-1} + \ldots + c_m = 0,$$

for some m and for some $c_i \in \mathfrak{a}^i$ for $1 \leq i \leq m$. In particular, for two monomial ideals $\mathfrak{a}$ and $\mathfrak{b}$ we have $P_{\mathfrak{a}} = P_{\mathfrak{b}}$ if and only if $\mathfrak{a}$ and $\mathfrak{b}$ have the same integral closure.

It is in general a difficult problem to relate the integral points in the Minkowski sum of two polytopes with the integral points in each of the polytopes. For example, it is an open problem to find conditions on two lattice polytopes P and Q such that

$$(P + Q) \cap N = (P \cap N) + (Q \cap N). \tag{4}$$

In particular, one would like to find conditions for a lattice polytope P and for $m \geq 2$ such that

$$mP \cap N = ((m-1)P \cap N) + (P \cap N). \tag{5}$$

For example, it is shown in [EW] that this is the case for every P if $m \geq \dim(P)$.

Example 4. Let $N = \mathbb{Z}^3$ and let P be the convex hull of $(0,0,0)$, $(0,1,1)$, $(1,0,1)$ and $(1,1,0)$. Show that $(1,1,1)$ lies in $2P \cap N$, but it can not be written as the sum of two elements in $P \cap N$.

Exercise. Let P be a full-dimensional lattice polytope and let σ be the cone over $P \times \{1\}$ in $V \times \mathbb{R}$. Consider the elements $v_1, \ldots, v_r$ of $P \cap N$ and let S be the semigroup generated by the $(v_i, 1)$. Show that the $(v_i, 1)$ give the Hilbert basis of S, and $\sigma \cap (N \times \mathbb{Z})$ is equal to S^{sat}.

Deduce that (5) holds for P and for all $m \geq 2$ if and only if S is saturated. Moreover, for every P, (5) holds if $m \gg 0$.

Problem. Let $\mathfrak{a} \subseteq k[t_1, \ldots, t_n]$ be a monomial ideal. Recall that by [Ho] the multiplier ideal of $\mathfrak{a}$ of exponent $\lambda > 0$ can be described as

$$\mathcal{I}(\mathfrak{a}^\lambda) = (t^v \mid v \in \mathbb{N}^n, v + e \in \mathrm{Int}(\lambda P_{\mathfrak{a}})),$$

where $e = (1, \ldots, 1)$. On the other hand, the multiplier ideal version of Skoda's Theorem (see [EL]) says that for $\lambda \geq n$ one has

$$\mathcal{I}(\mathfrak{a}^\lambda) = \mathfrak{a} \cdot \mathcal{I}(\mathfrak{a}^{\lambda-1}). \tag{6}$$

Is it possible to give a combinatorial proof of this equality for monomial ideals? Are there any connections with the above mentioned result of [EW]? Is there a common generalization of these two statements?

The polar polyhedron. For polyhedra, the analogue of the notion of dual cone is that of **polar polyhedron**. Suppose that P is a polyhedron in V containing the origin. The polar set is defined by

$$P^\circ := \{u \in U \mid \langle u, v \rangle \geq -1 \quad \text{for all} \quad v \in P\}.$$

If we write P as the Minkowski sum $P' + \sigma_0$ for some polytope P' and some polyhedral convex cone σ_0, then $P^\circ = (P')^\circ \cap \sigma_0^\vee$. In the definition of $(P')^\circ$ it is enough to put the condition only for v a vertex of P', so $(P')^\circ$ is a polyhedron. This implies that P° is also a polyhedron. Moreover, if P is rational, then so is P°.

Proposition 12. *Let P be a convex polyhedron containing the origin.*

1. *If σ is the closed cone over $P \times \{1\}$, then $\sigma^\vee$ is the closed cone over $P^\circ \times \{1\}$.*
2. *We have $(P^\circ)^\circ = P$.*
3. *The codimension of P° is equal to the largest dimension of an affine subspace contained in P.*

(4) *P° is a polytope if and only if the origin lies in the interior of P (so in particular* $\dim(P) = \dim(V)$*).*

(5) *If Q is a face of P not containing the origin, then*

$$Q^* := \{u \in P^\circ \mid \langle u, v\rangle = -1 \quad \textit{for all} \quad v \in Q\}$$

is a face of P° with $\dim(Q^*) = \dim(V) - 1 - \dim(Q)$. *This gives an inclusion-reversing bijection between the faces of P not containing the origin and those of P° with the same property. Moreover, we have $(Q^*)^* = Q$.*

Proof. Since σ is the closure of the cone generated by $P \times \{1\}$ we have

$$\sigma^\vee = \{(u, s) \mid \langle u, v\rangle + s \geq 0 \quad \text{for all} \quad v \in P\}.$$

On the other hand, the closed cone over $P^\circ \times \{1\}$ is

$$\{(u, s) \mid s \geq 0, \langle u, v\rangle + s \geq 0 \,\text{for all}\, v \in P\}.$$

In order to see that this is equal to $\sigma^\vee$, note that if $\langle u, v\rangle + s \geq 0$ for all v in P, this must hold in particular for $v = 0$, so $s \geq 0$.

Note that the origin in U lies in P°. From (1) and the fact that $(\sigma^\vee)^\vee = \sigma$ we deduce that $(P^\circ)^\circ = P$.

It follows from Proposition 5 that the codimension of $\sigma^\vee$ is equal to the codimension in $V \times \mathbb{R}$ of the largest linear subspace contained in σ. This subspace is $\sigma_0 \cap (-\sigma_0)$, where $\sigma_0 = \sigma \cap (V \times \{0\})$.

Recall that we may write P as $P' + \sigma_0$ for some polytope P', where we identify σ_0 with the corresponding cone in V. If P is the set of all v such that $\langle u_i, v\rangle \geq b_i$ for $1 \leq i \leq r$, then σ_0 is the set of all v such that $\langle u_i, v\rangle \geq 0$ for all i. It follows that if L is an affine subspace contained in P, then L is a translate of a linear subspace contained in σ_0. Since $\operatorname{codim}(\sigma^\vee) = \operatorname{codim}(P^\circ)$, we deduce (3).

If the origin lies in the interior of P, then P° is bounded, so it is a polytope. On the other hand, if P is bounded, then the origin lies in the interior of P°. Together with (2) this proves (4).

If Q is a face of P, let τ be the closed cone over $Q \times \{1\}$. We have

$$\sigma^\vee \cap \tau^\perp = \{(u, s) \in \sigma^\vee \mid \langle u, v\rangle + s = 0 \,\text{for all}\, v \in Q\},$$

so $(\sigma^\vee \cap \tau^\perp) \cap (U \times \{1\}) = Q^* \times \{1\}$. If we show that $\sigma^\vee \cap \tau^\perp$ is not contained in $U \times \{0\}$, then it follows that Q^* is associated to the face $\sigma^\vee \cap \tau^\perp$ by the correspondence in Lemma 11.

As Q is a face of P, we can find $b \in \mathbb{R}$ and $w \in U$ such that $\langle w, v\rangle \geq b$ for all $v \in P$ and $Q = P \cap \{v \mid \langle w, v\rangle = b\}$. Since the origin lies in P, we must have $b \leq 0$, while the fact that the origin does not lie in Q gives $b < 0$. Therefore $(w, -b)$ is in $(\sigma^\vee \cap \tau^\perp) \smallsetminus (U \times \{0\})$, so $\sigma^\vee \cap \tau^\perp$ is the closed cone over the face Q^* of P°.

Note that the origin in U does not lie in Q^*, so we may apply the previous discussion to Q^*. The assertions in (5) follow now from Proposition 5.

Remark 13. If P is a full-dimensional polytope containing the origin in its interior, we see that P° satisfies the same properties. The above proposition gives a bijection between the proper faces of P and those of P°.

For future reference, we include the following lemma.

Lemma 12. *Consider a nonempty polyhedron defined by*

$$P = \{v \in V \mid \langle u_i, v\rangle \geq b_i \text{ for } 1 \leq i \leq m\},$$

for some nonzero u_i in U and some b_i in $\mathbb{R}$. If σ is a cone in the normal fan Δ_P to P, then σ is generated by some of the u_i that satisfy $\langle u_i, v\rangle = b_i$ for some v in P.

Proof. Note that since any polyhedron is connected, if $\langle u_i, v\rangle > b_i$ for every v in P, then we may simply omit the equation $\langle u_i, v\rangle \geq b_i$ in the definition of P. It follows that after omitting some of the u_j, we may assume that for every i we have $\langle u_i, v\rangle = b_i$ for some v in P.

After possibly translating P, we may assume that it contains the origin, so $b_i \leq 0$ for every i. We reduce first to the case when the linear span W of P is V. Let C be the closed convex cone over $P \times \{1\}$ in $V \times \mathbb{R}$, so C is the dual of the cone generated by the $(u_i, -b_i)$.

The linear span of C is $W \times \mathbb{R}$, so the smallest face of $C^\vee$ is $W^\perp \times \{0\}$. Since $C^\vee$ is generated by the $(u_i, -b_i)$, it follows that $W^\perp$ is generated by those u_i that lie in $W^\perp$ (for which $b_i = 0$). Therefore we may replace V by W, so we may assume that $W = V$. In this case every cone σ in Δ_P is strongly convex, and in order to prove the proposition it is enough to consider the case when σ is a ray in Δ_P. In this case $\sigma = \sigma_Q$ for some facet Q of P. It follows from the description of P by equations that such a facet is of the form $P \cap \{v \mid \langle u_j, v\rangle = b_j\}$ for some j. Therefore u_j lies in σ_Q, and so it generates σ_Q, as required.

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