LECTURE 1. DEFINITIONS AND FIRST EXAMPLES

In this lecture we introduce the varieties that are rational or close to being rational and discuss some first easy examples.

1. Rational and unirational varieties: definitions

Let k be a fixed field. We will be mostly interested in the case when k is algebraically closed. However, for certain key arguments, it is important to have the definitions and some basic facts developed over arbitrary fields. We will also be mostly interested in the case when $\mathrm{char}(k) = 0$. Whenever this simplifies the arguments, we will restrict to this case.

A *variety* over k is a scheme of finite type over k, which is integral and separated.

Definition 1.1. A variety X over k is rational (over k) if it is birational to $\mathbf{P}^n_k$ (of course, in this case $n = \dim(X)$); equivalently, the function field $k(X)$ is isomorphic over k to the purely transcendental extension $k(t_1, \ldots, t_n)$.

It is clear from definition that in order to determine whether X is rational, we may replace X by any variety that is birational to X. For example, we may assume that X is affine. However, this is rarely useful, as most known techniques are of global nature. It is more useful to replace X by a projective model (after taking an affine open subset and taking the closure in a suitable projective case). When $\mathrm{char}(k) = 0$, it follows from Hironaka's theorem on resolution of singularities that we may replace X by a smooth, projective variety in the same birational equivalence class.

It is a classical (and usually very hard) problem to determine whether a given variety is rational. Typically, in order to prove that a variety is rational, one has to use explicit geometric constructions to give a birational map to $\mathbf{P}^n_k$. The particularly difficult part is proving that varieties are not rational: there are only partial methods for handling this and they all tend to involve subtle invariants.

Remark 1.2. If X is rational over k and K/k is a field extension, then

$$X_K := X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, K$$

is irreducible and rational over K (when considered with the reduced scheme structure). Indeed, by assumption we have an open, dense subset U of X such that U has an open embedding in $\mathbf{A}^n_k$. In this case U_K has an open embedding in $\mathbf{A}^n_K$ and therefore its closure in X_K is irreducible and rational over K. Moreover, since K/k is flat and the inclusion $U \hookrightarrow X$ is dominant, the same holds for $U_K \hookrightarrow X_K$, hence X_K is irreducible and rational. For an easy example such that X_K is rational over K, but X is not rational over k, see Example 1.7 below.

We will consider various other notations that make a variety be "close to rational". We begin with the definition of unirationality.

Definition 1.3. A variety X over k is unirational (over k) if there is a rational dominant map $f\colon \mathbf{P}_k^N \dashrightarrow X$ for some N; equivalently, there is a dominant morphism $Y \to X$, with Y a rational variety.

Remark 1.4. If X is unirational, with $\dim(X) = n$, then there is a rational dominant map $f\colon \mathbf{P}_k^n \dashrightarrow X$. We give an argument assuming that the field k is infinite (for the general case, see for example [Oja90, Proposition 1.1]). Let $f\colon \mathbf{P}_k^N \dashrightarrow X$ be as in the definition. It is clear that $N \geq n$. Suppose that f is defined on the open subset U of $\mathbf{P}_k^N$. If $N > n$, then consider a general point $x \in X$, such that $f^{-1}(x)$ has dimension $N - n > 0$. Since k is infinite, if $H \subset \mathbf{P}_k^N$ is a general hyperplane (defined over k), we have $H \cap U \neq \emptyset$ and $\dim(H \cap f^{-1}(x)) = N - n - 1$. In this case, the restriction of f to $U \cap H$ gives a dominant morphism $U \cap H \to X$ (if this is not dominant, then all its fibers have dimension $\geq (N-1) - (n-1) = N - n$, a contradiction). We thus have a rational dominant map $\mathbf{P}_k^{N-1} \dashrightarrow X$. By repeating this argument we obtain our assertion.

Remark 1.5. If X is a unirational variety over k, then for every field extension K/k, we have that X_K is irreducible and unirational over K (when considered with the reduced scheme structure). Indeed, if $f\colon Y \to X$ is a dominant morphism, with Y rational over k, then $Y_K \to X_K$ is dominant, and Y_K is irreducible and rational over K by Remark 1.2. Therefore X_K is irreducible and rational. A simple example of a non-unirational variety X over k such that X_K is unirational (even rational) over K, for some finite field extension K/k, is given in Example 1.7 below.

It is clear from definition that if X and Y are birational varieties, then X is unirational if and only if Y is unirational. It is also clear that if X is rational, then X is automatically unirational. A problem that generated a lot of activity over the years is the Lüroth problem: is the converse true? More precisely, is every unirational variety rational? We will see in this class that the answer is negative and the reasons for this come in different flavors.

As we will see shortly, the answer is positive in dimension 1. In dimension 2, we will see that the answer is positive in characteristic 0, but not in positive characteristic. On the other hand, in dimension 3, there are several examples of unirational varieties over fields of characteristic 0 that are not rational. The first such examples appeared around the same time, at the beginning of the 70s. Clemens and Griffiths showed in [CG72] that a smooth hypersurface of degree 3 in $\mathbf{P}^4$ is not rational (while such hypersurfaces have been known classically to be unirational). Iskovskikh and Manin showed in [IM71] that a smooth hypersurface of degree 4 in $\mathbf{P}^4$ is not rational (while Segre gave examples of such hypersurfaces that are unirational). Artin and Mumford then gave in [AM72] another example of a 3-fold (a conic bundle over a rational surface) that is unirational, but not rational. The plan is to discuss in detail, during this course, generalizations of the examples in [IM71] and [AM72].

An algebraic reformulation of the Lüroth problem is the following: given field extensions

$$k \subseteq K \subseteq k(t_1, \ldots, t_n),$$

where $t_1, \ldots, t_n$ are algebraically independent over k, does it follow that K is purely trancendental over k? Recall that every field subextension of a finitely generated field extension is automatically finitely generated (see, for example, [Oja90, Proposition 1.2]), hence K is automatically the function field of a variety over k.

Remark 1.6. If X is a unirational variety over an infinite field k, then $X(k) \neq \emptyset$: this follows since for every nonempty open subset U of $\mathbf{A}^n$, we have $U(k) \neq \emptyset$. The assertion also holds over a finite field when X is complete, but we do not prove it, since we will not need it. It is a consequence of Nishimura's lemma, which says that if $f\colon Y \dashrightarrow X$ is a rational map of varieties such that Y is smooth, X is complete, and $Y(k) \neq \emptyset$, then $X(k) \neq \emptyset$; see [RY00, Proposition A6] for a proof.

Example 1.7. It follows from the previous remark that the quadric $Q \subseteq \mathbf{P}^2_{\mathbf{R}}$ defined by $(x^2 + y^2 + z^2)$ is not rational (and not even unirational). On the other hand, it will follow from Example 3.8 below that after base-changing to $\mathbf{C}$, the quadric becomes rational.

We now define an intermediate notion between "rational" and "unirational".

Definition 1.8. A variety X over k is *stably rational* (over k) if $X \times \mathbf{P}^m_k$ is rational for some m.

It is clear from definition that if X and Y are birational varieties, then X is stably rational if and only if Y is. We also have the following implications:

$$\text{rational} \Rightarrow \text{stably rational} \Rightarrow \text{unirational}.$$

The first implication follows directly from definition, while the second implication follows from the fact that if $X \times \mathbf{P}^m_k$ is birational to $\mathbf{P}^{m+n}_k$, then we have a rational dominant map given by the composition

$$\mathbf{P}^{m+n}_k \dashrightarrow X \times \mathbf{P}^m_k \xrightarrow{p} X,$$

where p is the projection onto the first component.

It is known that both converses of the above implications fail: the Artin-Mumford example in [AM72] is unirational, but not stably rational, while Beauville, Colliot-Thélène, Sansuc, and Swinnerton-Dyer constructed in [BCSS85] an example of a non-rational complex 3-fold X such that $X \times \mathbf{P}^3$ is rational.

The following proposition shows that rationality, unirationality, and stable rationality for the extension to a larger field implies the same property over the original field, as long as this is algebraically closed.

Proposition 1.9. *Let X and Y be geometrically integral*[1] *varieties over a field k and let K/k be a field extension.*

[1] Recall that a scheme X over k is geometrically integral if X_K is integral for every field extension K/k. In fact, it is enough to require $X_{\overline{k}}$ be integral, where $\overline{k}$ is the algebraic closure of k.

i) *If there is a generically finite, dominant rational map* $f\colon X_K \dashrightarrow Y_K$ *of degree* d, *then there is a field* $k \subseteq k' \subseteq K$ *with* k'/k *finite, and a generically finite, dominant rational map* $g\colon X_{k'} \dashrightarrow Y_{k'}$ *of degree* d.

ii) *If* X_K *is rational (unirational, stably rational), then there is a finite field extension* k/k' *such that* $X_{k'}$ *is rational (resp., unirational, stably rational).*

Proof. The assertions in ii) follow from that in i) and the definitions of rational, unirational, and stably rational varieties. We thus only prove i). After replacing K by a suitable subfield finitely generated over k, such that f is defined over k', we may assume that K is finitely generated over k. Therefore there is a variety W over k such that $K = k(W)$. After replacing W by a suitable open subset, we may assume that $X \times W$ and $Y \times W$ are integral schemes, and that we have a generically finite, dominant, rational map $\varphi\colon X \times W \dashrightarrow Y \times W$ over W, of degree d. Therefore we can find open subsets $U \subseteq X \times W$ and $V \subseteq Y \times W$ such that φ is given by a finite flat morphism $U \to V$ of degree d. If $w \in W$ is a closed point that lies in the image of both U and V in W, and if $k' = k(w)$, then k'/k is finite and we obtain a finite flat morphism of degree d (in particular, dominant)

$$X_{k'} \supseteq U \times_W \mathrm{Spec}(k') \to V \times_W \mathrm{Spec}(k') \subseteq Y_{k'}.$$

□

2. The Lüroth problem in dimension 1

We show that for curves, the answer to the Lüroth problem is always positive. This is a classical result that goes back 140 years.

Theorem 2.1 (Lüroth, [Lur76])**.** *If* X *is a unirational variety over* k, *with* $\dim(X) = 1$, *then* X *is rational.*

Proof. We first give a geometric argument, under the assumption that k is algebraically closed, with $\mathrm{char}(k) = 0$. After replacing X by a projective model and taking its normalization, we may assume that X is smooth and projective. Let g be the genus of X. By assumption, we have a rational dominant map $f\colon \mathbf{P}^1 \dashrightarrow X$. Like every rational map between smooth, projective curves, this is in fact a morphism. Since we are in characteristic 0, the morphism f is separable, and we can thus apply the Hurwitz formula (see [Har77, Corollary IV.2.4]) to get

$$-2 = \deg(f) \cdot (2g - 2) + \deg(R),$$

where R is the ramification divisor. Since R is effective, we conclude that $g < 1$, hence $X \simeq \mathbf{P}^1$.

We now give an elementary algebraic argument, that is valid over any field, following [Oja90, Theorem 1.3]. We show that given any field extensions

$$k \subseteq K \subseteq k(t),$$

with t trancendental over k, and with $\mathrm{trdeg}_k(K) = 1$, we have $K \simeq k(x)$, where x is trancendental over k. Note that since we know that $\mathrm{trdeg}_k(K) = 1$, it is enough to find $a \in K$ such that $K = k(a)$.

Since $\mathrm{trdeg}_k(K) = 1 = \mathrm{trdeg}_k k(t)$, it follows that the extension $k(t)/K$ is algebraic. In particular, t is algebraic over K, and let

$$f(x) = x^n + a_1x^{n-1} + \ldots + a_n \in K[x]$$

be the minimal polynomial of t over K. Since t is trancendental over k, we can't have all a_i lying in k. Let i be such that $a_i \in K \smallsetminus k$. We will show that in this case $K = k(a_i)$, which as we have seen, gives the assertion in the theorem.

Since $a_i \in k(t)$, we can write $a_i = \frac{u(t)}{v(t)}$, where $u, v \in k[t]$ are relatively prime polynomials. The assumption that $a_i \not\in k$ implies that at least one of u and v has positive degree. Consider now the following polynomial

$$F(x) = u(x) - a_iv(x) \in k(a_i)[x] \subseteq K[x].$$

Since $F(t) = 0$, it follows that f divides F in $K[x]$, hence we can write

$$u(x) - a_iv(x) = f(x)g(x), \tag{1}$$

for some $g \in K[x]$. In order to complete the argument, it is enough to show that g is a constant polynomial: indeed, if this is the case, then t is the root of a degree n polynomial in $k(a_i)[x]$, hence $[k(t) : k(a_i)] \leq n = [k(t) : K]$, which implies $K = k(a_i)$.

After multiplying (1) by $v(t)$ and by the product of all denominators in the coefficients of f and g, we obtain an equality

$$c(t)\big(u(x)v(t) - v(x)u(t)\big) = f_1(x,t)g_1(x,t), \tag{2}$$

where $c(t) \in k[t]$ is nonzero and f_1 and g_1 are polynomials in $k[x,t]$ obtained by multiplying f and g, respectively, by nonzero elements in $k[t]$. Since $k[x,t]$ is a UFD, each prime factor of $c(t)$ divides either $f_1(x,t)$ or $g_1(x,t)$. After successively dividing by such factors, we obtain from (2) the equality

$$u(x)v(t) - v(x)u(t) = f_2(x,t)g_2(x,t), \tag{3}$$

where f_2 and g_2 are polynomials in $k[x,t]$, obtained by multiplying f and g, respectively, by nonzero elements in $k(t)$. It is clear that

$$\deg_t(u(x)v(t) - v(x)u(t)) \leq \max\{\deg(u(t)), \deg(v(t))\}. \tag{4}$$

On the other hand, we can write

$$f_2(x,t) = \sum_{j=0}^{n} b_j(t)x^{n-j}$$

and

$$\frac{b_i(t)}{b_0(t)} = a_i = \frac{u(t)}{v(t)}.$$

Since $u(t)$ and $v(t)$ are relatively prime, this implies that

$$\deg_t(f_2(x,t)) \geq \max\{\deg(u(t)), \deg(v(t))\}. \tag{5}$$

By combining (4) and (5), we conclude that $\deg_t(g_2(x,t)) = 0$, that is, $g_2 \in k[x]$. If $g_2 \notin k$, then it has a root γ in some algebraic closure of k. In this case, we deduce from (3)

$$u(\gamma)v(t) - v(\gamma)u(t) = 0.$$

Since $u(t)$ and $v(t)$ are relatively prime polynomials in $k[t]$, with at least one of them of positive degree, this implies that $u(\gamma) = 0 = v(\gamma)$. This however, contradicts the fact that $u(t)$ and $v(t)$ are relatively prime.

We thus conclude that $g_2 \in k$, which in turn implies that $g \in k(t) \cap K[x]$, hence $g \in K$. This completes the proof. $\square$

3. First examples of rational and non-rational varieties

The easiest way to prove that a variety is not rational is by finding invariants that agree on birational varieties and that take a different value on $\mathbf{P}_k^n$ and on the given variety. We begin with the simplest example of such invariants.

Proposition 3.1. *If $X \dashrightarrow Y$ is a separable, dominant, rational map between smooth, complete varieties of the same dimension, then*

$$h^0\big(X, (\Omega_X^q)^{\otimes m}\big) \geq h^0\big(Y, (\Omega_Y^q)^{\otimes m}\big) \quad \textit{for all} \quad q, m \geq 0.$$

In particular, if X and Y are birational smooth complete varieties, then

$$h^0\big(X, (\Omega_X^q)^{\otimes m}\big) = h^0\big(Y, (\Omega_Y^q)^{\otimes m}\big) \quad \textit{for all} \quad q, m \geq 0.$$

Proof. The argument is the same as the one in [Har77, II.8.19], showing that two birational smooth, complete varieties have the same geometric genus. Since X is normal and Y is complete, there is an open subset U of X with $\mathrm{codim}_X(X \smallsetminus U) \geq 2$ such that the rational map is given by a morphism $f\colon U \to Y$. Consider the morphism of vector bundles of the same rank

$$f^*\Omega_Y \to \Omega_U. \tag{6}$$

Since f is separable, it is generically smooth (hence étale). Therefore there is an open subset of U on which the morphism (6) is an isomorphism. Therefore the induced morphism of vector bundles

$$f^*(\Omega_Y^q)^{\otimes m} \to (\Omega_U^q)^{\otimes m}$$

is generically an isomorphism; in particular, it is injective. We deduce that we have an injective map on global sections

$$H^0\big(U, f^*(\Omega_Y^q)^{\otimes m}\big) \hookrightarrow H^0\big(U, (\Omega_U^q)^{\otimes m}\big). \tag{7}$$

On the other hand, since f is dominant, pulling-back sections gives an injective map

$$H^0\big(Y, (\Omega_Y^q)^{\otimes m}\big) \hookrightarrow H^0\big(U, f^*(\Omega_Y^q)^{\otimes m}\big). \tag{8}$$

Finally, since $\mathrm{codim}_X(X \smallsetminus U) \geq 2$, the variety X is normal, and $(\Omega_X^q)^{\otimes m}$ is locally free, restricting sections from X to U gives an isomorphism

$$H^0\big(X, (\Omega_X^q)^{\otimes m}\big) \simeq H^0\big(U, (\Omega_U^q)^{\otimes m}\big). \tag{9}$$

By combining (7), (8), and (9), we obtain the first assertion in the proposition. The second assertion is an immediate consequence. ☐

In particular, it follows from the proposition that the *plurigenera* of smooth, complete varieties are birational invariants. Recall that these are defined by $p_m(X) = h^0\big(X, \omega_X^{\otimes m}\big)$ for $m \geq 1$.

Remark 3.2. It is *not* true that the invariants $h^0(X, \omega_X^{-m})$ are birational invariants of smooth, projective varieties. For example, if $f\colon Y \to X$ is the blow-up of a smooth, projective surface X at a point P, with exceptional divisor E, then $\omega_Y \simeq f^*\omega_X \otimes \mathcal{O}_Y(E)$. Therefore

$$H^0\big(Y, \omega_Y^{-1}\big) \simeq H^0(X, \omega_X \otimes \mathcal{I}_P),$$

where $\mathcal{I}_P$ is the ideal defining P in X. For example, we see that if $X = \mathbf{P}^2$, then

$$h^0(X, \omega_X^{-1}) = 10 \quad \text{and} \quad h^0(Y, \omega_Y^{-1}) = 9.$$

Corollary 3.3. *If* $\mathrm{char}(k) = 0$ *and* $X \dashrightarrow Y$ *is a dominant rational map between smooth, projective varieties of the same dimension, then*

$$h^p(X, \mathcal{O}_X) \geq h^p(Y, \mathcal{O}_Y) \quad \textit{for all} \quad p \geq 0.$$

Without any restriction on the characteristic, if X and Y are birational smooth, projective varieties, then

$$h^p(X, \mathcal{O}_X) = h^p(Y, \mathcal{O}_Y) \quad \textit{for all} \quad p \geq 0.$$

Proof. It is a consequence of Hodge theory that for a smooth, projective variety X in characteristic 0, we have

$$h^p(X, \Omega_X^q) = h^q(X, \Omega_X^p) \quad \text{for all} \quad p, q \geq 0.$$

In particular, we have $h^p(X, \mathcal{O}_X) = h^0(X, \Omega_X^p)$ and the assertions in the corollary follow from Proposition 3.1. The fact that the second assertion also holds in positive characteristic is more subtle, and in fact this is a recent result, see [CR11]. ☐

Corollary 3.4. *If X is a smooth, projective variety which is rational, then*

$$h^0\big(X, (\Omega_X^q)^{\otimes m}\big) = 0 \quad \textit{for all} \quad m, q \geq 1, \ \textit{and}$$

$$h^p(X, \mathcal{O}_X) \quad \textit{for all} \quad p \geq 1.$$

Moreover, the same vanishings hold if X is just unirational when $\mathrm{char}(k) = 0$.

Proof. The assertions follow from Proposition 3.1 and Corollary 3.3 if we show that the vanishings hold when $X = \mathbf{P}^n$. The fact that the higher cohomology of the structure sheaf vanishes on projective space is well-known. Note now that it follows from the Euler sequence that we have an injective map

$$(\Omega_{\mathbf{P}^n}^q)^{\otimes m} \hookrightarrow \mathcal{E},$$

where $\mathcal{E}$ is isomorphic to a direct sum of line bundles isomorphic to $\mathcal{O}_{\mathbf{P}^n}(-mq)$. This clearly implies that

$$H^0\big(\mathbf{P}^n, (\Omega_{\mathbf{P}^n}^q)^{\otimes m}\big) = 0 \quad \text{for all} \quad m, q \geq 1.$$

☐

Corollary 3.5. *If $X \subseteq \mathbf{P}^n$ is a smooth subvariety that is a complete intersection of type $(d_1, \ldots, d_r)$, with $d_1 + \ldots + d_r \geq n+1$, then X is not rational. If* $\mathrm{char}(k) = 0$*, then X is not even unirational.*

Proof. It follows from the adjunction formula that
$$\omega_X \simeq \mathcal{O}_X(d_1 + \ldots + d_r - n - 1),$$
hence $h^0(X, \omega_X) \neq 0$. The assertions then follow from Corrolary 3.4. $\square$

Remark 3.6. The hypothesis that $\mathrm{char}(k) = 0$ is essential for getting that X is not unirational in the above corollary. In fact, in the case $\mathrm{char}(k) > 0$, we will discuss later examples due to Shioda of smooth surfaces in $\mathbf{P}_k^3$, of degree > 3, that are unirational. Similarly, the smoothness assumption in Corollary 3.5 can't be dropped, as shown by the following example.

Example 3.7. We show that if $X \subseteq \mathbf{P}_k^n$ is an integral hypersurface of degree $d \geq 2$ and if $P \in X(k)$ is such that $\mathrm{mult}_P(X) = d-1$, then X is rational. After a change of variable, we may assume that $P = (1, 0, \ldots, 0)$ and let $f \in k[x_0, \ldots, x_n]$ be a polynomial defining X. The affine variety $X \cap U \subseteq U = (x_0 \neq 0) \simeq \mathbf{A}^n$ is defined by the polynomial $g(x_1, \ldots, x_n) = f(1, x_1, \ldots, x_n)$. We can write
$$g = \sum_{i=0}^{d} g_i(x_1, \ldots, x_n),$$
with g_i homogeneous of degree i, in which case
$$\mathrm{mult}_P(X) = \min\{i \mid g_i \neq 0\}.$$
We thus see that $\mathrm{mult}_P(X) \leq d$ and equality holds if and only if $f \in k[x_1, \ldots, x_n]$, that is, X is the projective cone with vertex P over a hypersurface in $\mathbf{P}_k^{n-1}$. Of course, in this case we can't say anything about the rationality of X.

Suppose now that $\mathrm{mult}_P(X) = d-1$. The idea is that a general line through P will meet X at only one other point different from P. Since the lines through P in $\mathbf{P}_k^n$ are parametrized by $\mathbf{P}_k^{n-1}$, this gives a birational map between $\mathbf{P}_k^{n-1}$ and X. We now explain this in detail.

The hypothesis that $\mathrm{mult}_P(X) = d-1$ says that we have $g = g_{d-1} + g_d$, with $g_{d-1} \neq 0$. Note that we also have $g_d \neq 0$: otherwise $f = x_0 g(x_1, \ldots, x_n)$, hence it is not irreducible. Given $\lambda = (\lambda_1, \ldots, \lambda_n) \in k^n \smallsetminus \{0\}$ and $t \in k \smallsetminus \{0\}$, we have
$$g(t\lambda) = t^{d-1}\big(g_{d-1}(\lambda) + t g_d(\lambda)\big).$$
It follows that if $g_d(\lambda) \neq 0$, then the point $t\lambda \in U(k)$ lies in $X(k)$ if and only if $t = -\frac{g_{d-1}(\lambda)}{g_d(\lambda)}$. We now define
$$\varphi\colon \mathbf{P}_k^{n-1} \smallsetminus \{\lambda\colon g_d(\lambda) \neq 0\} \to U \cap X \quad \text{and} \quad \psi\colon (U \cap X) \smallsetminus \{P\} \to \mathbf{P}_k^{n-1}$$
given by
$$\varphi(\lambda_1, \ldots, \lambda_n) = \left(-\frac{g_{d-1}(\lambda)\lambda_1}{g_d(\lambda)}, \ldots, -\frac{g_{d-1}(\lambda)\lambda_n}{g_d(\lambda)}\right) \quad \text{and}$$
$$\psi(u_1, \ldots, u_n) = (u_1, \ldots, u_n).$$

It is straightforward to see that these give inverse rational maps between $\mathbf{P}_k^{n-1}$ and X.

Example 3.8. An important special case of the previous example is that if $X \subseteq \mathbf{P}_k^n$ is an integral quadric such that $X_{\mathrm{sm}}(k) \neq \emptyset$, then X is rational (note that conversely, if X is rational, or just unirational, then $X_{\mathrm{sm}}(k) \neq \emptyset$ by Remark 1.6). In particular, if k is algebraically closed, then every integral quadric over k is rational.

4. Uniruled varieties

While this will not feature much in what follows, we now introduce a weaker property than unirationality, but which is easier to study and features prominently in birational geometry.

Definition 4.1. A variety X over k of dimension n is uniruled (over k) if there is a variety Z over k of dimension $n-1$ and a rational dominant map

$$Z \times \mathbf{P}_k^1 \dashrightarrow X.$$

Again, it is clear from definition that if X and Y are birational, then X is uniruled if and only if Y is. If X is a point, then X does not satisfy the above definition, but if $\dim(X) > 0$, then every unirational variety is uniruled. Indeed, if $\dim(X) = n$ and we have a rational, dominant map $\mathbf{P}_k^n \dashrightarrow X$, then we obtain a rational, dominant map $\mathbf{P}_k^{n-1} \times \mathbf{P}_k^1 \dashrightarrow X$, using the fact that $\mathbf{P}_k^n$ and $\mathbf{P}_k^{n-1} \times \mathbf{P}_k^1$ are birational.

Remark 4.2. The property of being uniruled is much weaker than that of being unirational. For example, if C is an elliptic curve, then $C \times \mathbf{P}_k^1$ is clearly uniruled, but it is not unirational: otherwise, using the projection $C \times \mathbf{P}_k^1 \to C$ we would obtain that C is unirational, hence rational by Theorem 2.1.

In characteristic 0, the examples in Corollary 3.5 fail not only to be unirational, but also to be uniruled. This follows from the argument in the proof of Corollary 3.5 and the following easy result.

Proposition 4.3. *If X is a smooth, complete variety over a field k of characteristic 0, then all plurigenera $p_m(X) = h^0(X, \omega_X^{\otimes m})$, for $m \geq 1$, are 0.*

Proof. By assumption, we have a dominant rational map $Y \times \mathbf{P}_k^1 \dashrightarrow X$, where $\dim(Y) = \dim(X) - 1$. Since we are in characteristic 0, after replacing Y by a birational variety, we may assume that Y is smooth and projective. We then deduce from Proposition 3.1 that

$$p_m(X) \leq p_m(Y \times \mathbf{P}_k^1) \quad \text{for all} \quad m \geq 1,$$

hence it is enough to prove the assertion when $X = Y \times \mathbf{P}_k^1$. If $p\colon Y \times \mathbf{P}_k^1 \to Y$ and $q\colon Y \times \mathbf{P}_k^1 \to \mathbf{P}_k^1$ are the two projections, then

$$\omega_{Y \times \mathbf{P}_k^1} \simeq p^*\omega_Y \otimes q^*\mathcal{O}_{\mathbf{P}_k^1}(-2)$$

and it follows from the Künneth formula that

$$H^0(Y \times \mathbf{P}_k^1, \omega_{Y \times \mathbf{P}_k^1}^{\otimes m}) \simeq H^0\big(Y, \omega_Y^{\otimes m}\big) \otimes_k H^0\big(\mathbf{P}_k^1, \mathcal{O}_{\mathbf{P}_k^1}(-2m)\big) = 0.$$

$\square$

Remark 4.4. It is a major open problem in birational geometry that the converse also holds: if X is a smooth, projective variety over a field of characteristic 0, such that $p_m(X) = 0$ for all $m \geq 1$, then X is uniruled.

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