Chapter 13: Ehrhart polynomials

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Counting integral points via quasipolynomials. Let P be a rational polytope in $M_{\mathbb{R}}$ of dimension r. For a positive integer m, we put

$$f_P(m) := \mathrm{Card}(mP \cap M).$$

Our first goal is to prove that f_P is a **quasipolynomial**, i.e. there is a positive integer ℓ and polynomials Q_i for $i = 1, \ldots, \ell$ such that $f_P(m) = Q_i(m)$ if $m \equiv i \pmod{\ell}$. If this is the case, the **degree** of the quasipolynomial is the maximum of the degrees of the Q_i. The smallest ℓ as above is called the **period** of the quasipolynomial.

Proposition 1. *If P is a rational polytope of dimension r, then f_P is a quasipolynomial of degree r. In fact, each of the polynomials involved is either zero or has degree r. Moreover, if ℓ is such that ℓP is a lattice polytope, then the period of f_P divides ℓ.*

The proof of the proposition is based on the following general fact due to Snapper (for a different proof see [St]). If X is a complete scheme over k of dimension r, $\mathcal{F}$ is a coherent sheaf on X and $\mathcal{L}$ is a line bundle on X, then there is a polynomial $Q(t)$ of degree at most r, such that $Q(m) = \chi(X, \mathcal{F} \otimes \mathcal{L}^m)$ for every m in $\mathbb{Z}$. Moreover, it follows from the cohomological definition of intersection numbers (see [Kl]) that the coefficient of t^r in Q is $\mathrm{rk}(\mathcal{F}) \cdot (\mathcal{L}^n)/n!$.

Proof of Proposition 1. Suppose first that there is a lattice point in the affine span of P. After suitably translating P, we may assume that this point is the origin. In this case, P defines an ample divisor D on the toric variety X corresponding to the normal fan Δ_P of P, such that $P_D = P$. Recall that $\dim(X) = r$. Therefore we have

$$f_P(m) = h^0(X, \mathcal{O}(\lfloor D \rfloor)) = \chi(X, \mathcal{O}(\lfloor D \rfloor)),$$

where the second equality follows from Corollary 1 in Chapter 6.

Let ℓ be a positive integer such that ℓP is a lattice polytope (or equivalently, ℓD is Cartier). If $\mathcal{L} = \mathcal{O}(\ell D)$ and $m \equiv i \pmod{\ell}$, then

$$f_P(m) = \chi(X, \mathcal{O}(\lfloor iD \rfloor) \otimes \mathcal{L}^{(m-i)/\ell}).$$

Note that $\mathrm{rk}(\mathcal{O}(\lfloor iD \rfloor) = 1$ and $(\mathcal{L}^n)$ is positive since $\mathcal{L}$ is ample, so the assertion in the proposition follows from the above result of Snapper.

We consider now the case of an arbitrary rational polytope P. Since P is rational, there is a positive integer ℓ' such that the affine span of $\ell' P$ contains a lattice point, and we assume that ℓ' is the smallest such integer. For those m which are divisible by ℓ' we may apply the previous argument. On the other hand, if m is not divisible by ℓ', then there is no lattice point in the affine span of mP. Indeed, if u is such a point and w in M is in the affine span of $\ell'P$, let $1 \leq i \leq \ell' - 1$ be such that $m \equiv i \pmod{\ell'}$. We see that $u - \frac{m-i}{\ell'}w$ is a lattice point in the affine span of iP, so we get a contradiction.

In particular, we have $f_P(m) = 0$ whenever m is not divisible by ℓ', which completes the proof of the proposition.

In particular, it follows from the above proposition that if P is a lattice polytope, then f_P is a polynomial of degree r, called the **Ehrhart polynomial** of P.

Exercise 1. Show that for every nonzero polynomial which appears in the description of f_P as a quasipolynomial, the coefficient of t^r is $\mathrm{vol}_r(P)$.

A basic fact about Ehrhart polynomials is the following **reciprocity** result. The proof below generalizes the one given in [Da] in the case of a lattice polytope. We refer to [St] for the combinatorial side of the story.

Since f_P is a polynomial funmction on congruence classes modulo a certain positive integer, it can be extended in a unique way to all integers. Note that it follows from the proof of Proposition 1 that $f_P(0) = 1$.

Corollary 1. *If P is a rational polytope of dimension r, then for every positive integer m we have*

$$\mathrm{Card}(\mathrm{Int}(mP) \cap M) = (-1)^r f_P(-m).$$

Proof. It follows from the proof of Proposition 1 that if ℓ' is the smallest positive integer such that the affine span of $\ell'P$ contains a lattice point, then $f_P(-m)$ is zero whenever m is not divisible by ℓ'. On the other hand, in this case there is no lattice point in the interior of mP. Therefore it is enough to consider those m which are divisible by ℓ', which reduces the assertion to the case when there is a lattice point (which we may assume to be the origin) in the affine span of P.

In this case we have seen that there is an r-dimensional toric variety X and an ample $\mathbb{Q}$-T-divisor D on X such that $f_P(m) = \chi(X, \mathcal{O}(\lfloor mD \rfloor))$ for every positive integer m. Since we have extended f_P as a quasipolynomial to $\mathbb{Z}$, it follows that we have also $f_P(-m) = \chi(X, \mathcal{O}(\lfloor -mD \rfloor))$. Using Proposition 2 and Corollary 5 in Chapter 6, we deduce

$$f_P(-m) = (-1)^r \chi(X, \mathcal{O}(K_X + \lceil mD \rceil)) = (-1)^r h^0(X, \mathcal{O}(K_X + \lceil mD \rceil)),$$

so the assertion follows from Proposition 3 in Chapter 6..

The h-vector of a lattice polytope. We discuss now some further properties of Ehrhart polynomials of lattice polytopes. Since $f_P(m)$ is an integer for every positive integer m, it follows that $f_P(m) = \sum_{i=0}^{r} a_i\binom{i+m}{i}$ for (uniquely determined) integers $a_0, \ldots, a_r$ (see, for example, Lemma 4.1.4 in [BH]). Note that by Exercise 1 we have $a_r = r! \cdot \mathrm{vol}_r(P)$.

We form the generating series

$$F_P(t) = \sum_{m\in\mathbb{N}} f_P(m)t^m = \sum_{m\in\mathbb{N}}\sum_{i=0}^{r} a_i\binom{i+m}{i}t^m$$

$$= \sum_{i=0}^{r} a_i \cdot \sum_{m\in\mathbb{N}}\binom{i+m}{i}t^m = \sum_{i=0}^{r}\frac{a_i}{(1-t)^{i+1}},$$

so we can write

$$F_P(t) = \frac{\sum_{i=0}^{r} h_i t^i}{(1-t)^{r+1}}.$$

Note that we have $\sum_{i=0}^{r} a_i(1-t)^{r-i} = \sum_{i=0}^{r} h_i t^i$, so all h_i are integers. We call $(h_0, \ldots, h_r)$ the **h-vector** of P.

Consider also $\widetilde{F}_P(t) = \sum_{m\geq 1} f_P(-m)t^m$. We have

$$\widetilde{F}_P(t) = \sum_{i=0}^{n}(-1)^i a_i \sum_{m\geq i+1}\binom{m-1}{i}t^m$$

$$= \sum_{i=0}^{n}(-1)^i a_i \sum_{p\in\mathbb{N}}\binom{p+i}{i}t^{p+i+1} = \sum_{i=0}^{n}(-1)^i a_i\frac{t^{i+1}}{(1-t)^{i+1}} = -F_P(t^{-1}).$$

Therefore we have

$$(-1)^r\widetilde{F}_P(t) = \frac{\sum_{i=0}^{r} h_i t^{r-i+1}}{(1-t)^{r+1}}. \tag{1}$$

Lemma 1. *Let P be an r-dimensional lattice polytope.*

1. $\sum_{i=0}^{r} h_i = r! \cdot \mathrm{vol}_r(P)$.
2. $h_1 = \mathrm{Card}(P \cap M) - (r+1)$.
3. $h_r = \mathrm{Card}(\mathrm{Int}(P) \cap M)$.
4. *More generally, the largest d such that h_d is nonzero is equal to* $r + 1 - \min\{q \in \mathbb{N}^* \mid \mathrm{Int}(qP) \cap M \neq \emptyset\}$.

Proof. The first assertion follows since $\sum_{i=0}^{r} h_i = a_r$. Note that from the identity

$$(1-t)^{r+1} \cdot \sum_{m\geq 0} f_P(m)t^m = \sum_{i=0}^{r} h_i t^i$$

we can determine the h_i in terms of the values of $f_P(m)$ for $m \in \mathbb{N}$. In particular, considering the coefficient of t we deduce $h_1 = f_P(1) - (r+1)f_P(0)$, which gives (2).

Similarly, using (1) we can deduce formulas for the h_i in terms of $f_P(-m)$ for $m \in \mathbb{Z}_{<0}$. In particular, we have $h_r = (-1)^r f_P(-1) = \mathrm{Card}(\mathrm{Int}(P) \cap M)$. Moreover, we see that $h_i = 0$ for all $i > d$ if and only if $f_P(-i) = 0$ for every $i < r + 1 - d$, so we get (4).

Lemma 2. *If P is a lattice polytope of dimension r, then*

$$f_P(m) = \sum_Q (-1)^{\dim(Q)} f_Q(-m),$$

where the sum is over all faces Q of P (we assume Q nonempty, but possibly equal to P).

Proof. It is enough to show that we have this equality for all positive integers m. Every lattice point in mP lies in the relative interior of precisely one face Q of P. Since the number of lattice points in the interior of mQ is $(-1)^{\dim(Q)} f_Q(-m)$, we deduce the equality in the lemma.

Remark 1. Let σ be the cone over $P \times \{1\}$ in $M_{\mathbb{R}} \times \mathbb{R}$. Note that by definition, F_P is the Hilbert series of the ring $R = k[\sigma \cap (M \times \mathbb{Z})]$, where $\deg(\chi^{(u,m)}) = m$. We see that R is normal since it is the semigroup algebra of a saturated semigroup, and it is Cohen-Macaulay by Theorem 1 in Chapter 3.

Proposition 2. (Stanley) *With the above notation, all h_i are nonnegative.*

Proof. This is a particular case of a more general result about the Hilbert series of a Cohen-Macaulay graded ring (see Corollary 4.1.10 in [BH]). We recall the proof, referring for the basic facts on depth theory to [BH].

With R as in Remark 1, note that $\dim(R) = r + 1$. Moreover, R is finite over $k[R_1]$ (it is the normalization of $k[R_1]$ by the exercise at the bottom of p. 16 in Chapter 2). Since R is Cohen-Macaulay, the depth of R with respect to the maximal ideal generated by R_1 in $k[R_1]$ is $r+1$. Our base field k is infinite, so we may find $x_1, \ldots, x_{r+1}$ in R_1 which form a regular sequence in R. In this case the Hilbert series F_R and F_A of R and $A = R/(x_1, \ldots, x_{r+1})$, respectively, are related by $F_R(t) = F_A(t)/(1-t)^{r+1}$. Therefore $h_i = \dim_k(A_i)$, so it is nonnegative.

In general, one can describe when a Cohen-Macaulay graded ring is Gorenstein in terms of its Hilbert series (see Corollary 4.4.6 in [BH]). In our case, we have the following equivalent characterizations. Note that since P is a lattice polytope, in order to compute its Ehrhart polynomial, there is no loss of generality in assuming that $\dim(P) = \mathrm{rk}(M)$ (it is enough to consider a suitable translate of P in the linear space it spans).

Proposition 3. (Stanley, Hibi) *Let P be an r-dimensional lattice polytope in M, where $\mathrm{rk}(M) = r$. If d is the largest i such that h_i is nonzero and $\ell = r + 1 - d$, let w be a lattice point in the interior of ℓP. The following are equivalent:*

(1) *For every i we have $h_i = h_{d-i}$.*
(2) *For every m in $\mathbb{Z}$ we have $f_P(-m) = (-1)^r f_P(m - \ell)$.*
(3) $F_P(t^{-1}) = (-1)^{r+1} t^{\ell} F_P(t)$.
(4) *For every integer $m \geq \ell$, we have*

$$(\mathrm{Int}(mP) \cap M) - w = (m - \ell)P \cap M.$$

(5) *ℓP is a translate of a reflexive polyhedron.*
(6) *If σ is the cone over $P \times \{1\}$ in $M_{\mathbb{R}} \times \mathbb{R}$, then the ring $R = k[\sigma \cap (M \times \mathbb{Z})]$ is Gorenstein.*

Proof. Since $F_P(t) = \frac{1}{(1-t)^{r+1}} \sum_{i=0}^{d} h_i t^i$, we see that

$$F_P(t^{-1}) = \frac{(-1)^{r+1}}{(1-t)^{r+1}} \cdot \sum_{i=0}^{r} h_i t^{r+1-i}.$$

This gives (1)$\Leftrightarrow$(3). Since $\widetilde{F}_P = -F_P(t^{-1})$, we get (2)$\Leftrightarrow$(3).

Note that since w lies in the interior of ℓP, we always have an inclusion

$$(m - \ell)P \cap M \subseteq (\mathrm{Int}(mP) \cap M) - w. \tag{2}$$

Therefore this is an equality if and only if the two sets have the same cardinality, so we get (2)$\Leftrightarrow$(4) by Corollary 1.

We show now (4)$\Leftrightarrow$(6). Since R is Cohen-Macaulay, R is Gorenstein if and only the dualizing module ω_R is free of rank one. By Remark 6 in Chapter 3, $\omega_R = \oplus k \cdot \chi^u$, the direct sum being over those u in the interior of σ. Since w lies in the interior of ℓP, we deduce that $\chi^{(w,\ell)}$ is an element of ω_R. It is easy to see that ω_R is free of rank one if and only if it is generated by $\chi^{(w,\ell)}$. On the other hand, this assertion is equivalent with the fact that for every positive integer $m \geq \ell$, if u lies in $\mathrm{Int}(mP) \cap M$, then $u - w$ is in $(m - \ell)P \cap M$, which is precisely the condition that (2) is an equality. Therefore we have (4)$\Leftrightarrow$(6).

Note that by Proposition 13 in Chapter 6, if $X = X_P$ and $D = \ell D_P$, then $\mathcal{O}(D) \simeq \mathcal{O}(-K_X)$ if and only if ℓP is a translate of a reflexive polyhedron.

Condition (2) says that $(-1)^r\chi(X, \mathcal{O}(-mD)) = \chi(X, \mathcal{O}((m-\ell)D))$ for every integer m. Since f_P is a polynomial, it is enough to put this condition only for $m \geq \ell$. In this case, since D is ample the vanishing theorems and Serre duality give

$$\chi(X, \mathcal{O}(-mD)) = (-1)^n h^n(X, \mathcal{O}(-mD)) = (-1)^n h^0(X, \mathcal{O}(K_X+mD)),$$

$$\chi(X, \mathcal{O}((m-\ell)D)) = h^0(X, \mathcal{O}((m-\ell)D)).$$

Therefore we clearly have (6)⇒(2).

On the other hand, condition (2) for $m = \ell$ gives $h^0(X, \mathcal{O}(K_X + \ell D)) = 1$. This gives embeddings

$$j_m \colon \mathcal{O}((m-\ell)D) \hookrightarrow \mathcal{O}(K_X + mD)$$

and we see that (2) is equivalent with the fact that for every $m \geq \ell$ the morphism induced by j_m between the spaces of sections is an isomorphism. Since D is ample, it follows that if m is large enough, then both $\mathcal{O}((m-\ell)D)$ and $\mathcal{O}(K_X + mD)$ are generated by global sections. Hence (2) implies that j_m is an isomorphism, so (2)⇔(6). This completes the proof of the proposition.

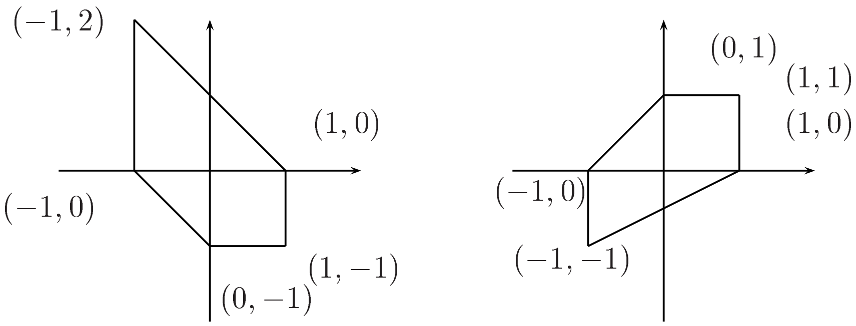
From the above proposition and Lemma 2, we get the following corollary.

Corollary 2. *If P is a reflexive polytope, then*

$$f_P(m) - f_P(m-1) = \sum_Q (-1)^{\dim(Q)} f_Q(-m).$$

Example 1. If $e_1, \ldots, e_n$ are the unit vectors in $M = \mathbb{Z}^n$, let P be the convex hull of $\{0, e_1, \ldots, e_n\}$. The toric variety defined by the normal fan Δ_P is $\mathbb{P}^n$ and the line bundle corresponding to P is $\mathcal{O}(1)$. It follows that $f_P(m) = \binom{m+n}{n}$, and therefore $h_0 = 1$ and $h_i = 0$ for $i \geq 1$.

Example 2. Let P and Q be the following plane polytopes



$P,\ X_P \simeq \mathrm{Bl}_{x_1,x_2}\mathbb{P}^2$ $\qquad$ $Q = P^\circ$

It follows from definition that the Ehrhart polynomial of P is given by $f_P(m) = \frac{7}{2}m^2 + \frac{7}{2}m + 1$, so the h-vector is $(1, 5, 1)$. For Q, the Ehrhart polynomial is given by $f_Q(m) = \frac{5}{2}m^2 + \frac{5}{2}m + 1$, and we get the corresponding h-vector $(1, 3, 1)$.

Example 3. Let M be the set of $n \times n$ matrices with integer coefficients, and let $P \subseteq M_{\mathbb{R}}$ be the set of matrices with nonnegative entries, having the sum of each row and column equal to one. The vertices of P are the permutation matrices. Note that $f_P(m)$ is the number of magic squares of sum m.

It is easy to see that $\dim(P) = (n-1)^2$. Stanley proved that P satisfies the equivalent conditions in Proposition 3, with $d = (n-1)(n-2)$ (the unique interior lattice point in nP is the matrix with all entries equal to one). Stanley further conjectured that the h-vector of P is unimodal: it satisfies $h_0 \leq h_1 \leq \ldots \leq h_{\lfloor d/2 \rfloor}$. A proof of this conjecture was announced in [At].

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