# Homework Set 6

Please, try to do all of the following problems. Solutions to five of them are due on Monday March 6.

**Problem 1**. Let $X$ be a scheme. Show that the set of isomorphism classes of locally free rank one sheaves on $X$ becomes an abelian group under the the tensor product. It is called the *Picard group* of $X$ and it is denoted by $\mathrm{Pic}(X)$.

**Problem 2**. The usual operations in multilinear algebra have a correspondent for $\mathcal{O}_X$-modules. Let us consider for example the case of symmetric powers. Suppose that $(X, \mathcal{O}_X)$ is a ringed space and that $\mathcal{F}$ an $\mathcal{O}_X$-module.

(1) The $m$th symmetric power $\mathrm{Sym}^m(\mathcal{F})$ is the sheaf associated to the presheaf
$$U \to \mathrm{Sym}^m_{\mathcal{O}_X(U)}(\mathcal{F}(U)).$$
Show that for every $x \in X$ there is a canonical isomorphism
$$\mathrm{Sym}^m(\mathcal{F})_x \simeq \mathrm{Sym}^m_{\mathcal{O}_{X,x}}(\mathcal{F}_x).$$
(2) Show that $\mathrm{Sym}(\mathcal{F}) := \oplus_m \mathrm{Sym}^m(\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-algebras.
(3) Show that if $X$ is a scheme and $\mathcal{F}$ is quasi(coherent), then so is $\mathrm{Sym}^m\mathcal{F}$. Moreover, $\mathrm{Sym}(\mathcal{F})$ is a quasicoherent sheaf of $\mathcal{O}_X$-algebras.
(4) Show that if $\mathcal{F} \simeq \mathcal{O}_X^n$, then $\mathrm{Sym}(\mathcal{F}) \simeq \mathcal{O}_X[T_1, \ldots, T_n]$.

For more on this and similar constructions, see Exercise II.5.16 in Hartshorne.

**Problem 3**. This is Problem II.5.18 from Hartshorne (describing the equivalence between vector bundles and locally free sheaves).

**Problem 4**. Let $X$ be a topological space. Show that if
$$0 \to \mathcal{F}' \xrightarrow{u} \mathcal{F} \xrightarrow{v} \mathcal{F}'' \to 0$$
is an exact sequence of sheaves of abelian groups on $X$ such that $\mathcal{F}'$ is flasque, then the following sequence is exact
$$0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0.$$
(Hint: given $s'' \in \Gamma(X, \mathcal{F}'')$, consider the set
$$\{(U, s) \mid U \,\mathrm{open\,in}\, X, s \in \mathcal{F}(U), v(s) = s''|_U\}$$
ordered by $(U, s) \leq (V, t)$ if $U \subseteq V$ and $t|_U = s$. Show that this set has a maximal element and that if $\mathcal{F}'$ is flasque and $(V, t)$ is maximal, then $V = X$).

**Problem 5**. Let $(X, \mathcal{O}_X)$ be a ringed space.

(1) If $U$ is an open subset of $X$ and $i \colon U \to X$ the inclusion, show that there is a sub-$\mathcal{O}_X$-module $i_!(\mathcal{O}_U)$ of $\mathcal{O}_X$ such that $i_!(\mathcal{O}_U)(V) = \mathcal{O}_X(V)$ if $V \subseteq U$, and $i_!(\mathcal{O}_U)(V) = 0$, otherwise.

(2) Show that if $x \in U$, then we have a canonical isomorphism $\mathcal{O}_{U,x} \simeq i_!(\mathcal{O}_U)_x$ and if $x \notin U$, then $i_!(\mathcal{O}_U)_x = 0$. The sheaf $i_!(\mathcal{O}_U)$ is called the *extension by zero* of $\mathcal{O}_U$.
(3) Show that for every $\mathcal{O}_X$-module $\mathcal{F}$, we have a canonical isomorphism
$$\mathrm{Hom}_{\mathcal{O}_X}(i_!\mathcal{O}_U, \mathcal{F}) \simeq \mathcal{F}(U).$$
(4) Deduce that if $\mathcal{I}$ is an injective $\mathcal{O}_X$-module, then it is flasque.
(5) Give an example of a scheme $X$ and an open subset $U$ such that $i_!\mathcal{O}_U$ is not quasicoherent.

**Problem 6**. This is Problem III 4.4 from Hartshorne (showing that for arbitrary sheaf of abelian groups $\mathcal{F}$ on a topological space $X$, there is an isomorphism of $H^1(X, \mathcal{F})$ with the inductive limit of the groups $H^1(\mathcal{U}; \mathcal{F})$, with $\mathcal{U}$ running over all open covers of $X$).

**Problem 7**. This is Problem III 4.5 from Hartshorne (showing that there is an isomorphism of abelian groups between $\mathrm{Pic}(X)$ and $H^1(X, \mathcal{O}_X^*)$.

**Problem 8**. This is Problem III 4.6 from Hartshorne (it relates the Picard group of a scheme $X$ with that of a closed subscheme defined by a sheaf of ideals $\mathcal{I}$ such that $\mathcal{I}^2 = 0$).