Homework Set 4

Please, try to do all of the following problems. Solutions to three of them are due on Monday February 13.

Problem 1. Let $f,\, g\colon X \to Y$ be two morphisms of schemes, with X reduced and Y separated. Show that if there is an open dense subset U of X such that $f|_U = g|_U$, then $f = g$ (where equality means equality as morphisms of schemes). Show that this is not necessarily true if X is not reduced or if Y is not separated.

Problem 2. Show that the class of proper morphisms is closed under composition and base change.

Problem 3. Show that if A is an arbitrary ring, then for every $m, n \geq 1$ there is a closed immersion

$$\mathbb{P}_A^{m-1} \times \mathbb{P}_A^{n-1} \hookrightarrow \mathbb{P}_A^{mn-1}$$

(the Segre embedding).

Problem 4. Use the previous problem to show that the classes of projective and quasiprojective morphisms are closed under compositions.

Problem 5. Let $\phi\colon R \to S$ be a graded morphism of graded rings (i.e. $\phi(R_m) \subseteq S_m$ for every $m \in \mathbb{N}$).

(i) Show that if $U = \{q \in \operatorname{Proj} S \mid \phi^{-1}(q) \not\supseteq R_+\}$ (where $R_+ = \oplus_{m>0} R_m$), then U is open in $\operatorname{Proj} S$ and ϕ induces a scheme morphism $f\colon U \to \operatorname{Proj} R$.

(ii) Show that if ϕ is surjective, then $U = \operatorname{Proj} S$ and f is a closed immersion.