APPENDIX 2: AN INTRODUCTION TO ÉTALE COHOMOLOGY AND THE BRAUER GROUP

In this appendix we review some basic facts about étale cohomology, give the definition of the (cohomological) Brauer group, and discuss those properties of this group that we need.

1. Faithfully flat descent

In this section we collect a few statements are are part of *descent theory*. For proofs and further discussions, we refer to [Mil80, p. 16-19]. Recall that a morphism of schemes $f\colon X \to Y$ is *faithfully flat* if it is flat and surjective. If both X and Y are affine schemes, then the morphism is faithfully flat if and only if $\mathcal{O}(X)$ is faithfully flat as an $\mathcal{O}(Y)$-module, that is, a complex

$$M' \to M \to M''$$

of $\mathcal{O}(Y)$-modules is exact if and only if

$$M' \otimes_{\mathcal{O}(Y)} \mathcal{O}(X) \to M \otimes_{\mathcal{O}(Y)} \mathcal{O}(X) \to M'' \otimes_{\mathcal{O}(Y)} \mathcal{O}(X)$$

is exact.

The following statement is not hard to prove.

Proposition 1.1. *If $A \to B$ is a faithfully flat ring homomorphism, then for every A-module M, we have an exact sequence*

$$0 \to M \xrightarrow{i} M \otimes_A B \xrightarrow{d} M \otimes_A (B \otimes_A B),$$

where $i(m) = m \otimes 1$ and $d(m \otimes b) = m \otimes b \otimes 1 - m \otimes 1 \otimes b$.

This proposition is used to deduce that every faithfully flat, finite type morphism of schemes $f\colon X \to Y$ is a strict epimorphism, as described below.

Proposition 1.2. *If $f\colon X \to Y$ is a finite type, faithfully flat morphism of schemes, then for every scheme Z, the induced map $\mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ is the equalizer of the two maps $\mathrm{Hom}(X, Z) \to \mathrm{Hom}(X \times_Y X, Z)$ induced by the two projections.*

Another consequence of Proposition 1.1 is the following description of quasi-coherent sheaves on the image of a faithfully flat morphism.

Proposition 1.3. *If $f\colon X \to Y$ is a quasi-compact, faithfully flat morphism of schemes, then giving a quasi-coherent sheaf $\mathcal{M}$ on Y is equivalent to giving a quasi-coherent sheaf $\mathcal{N}$ on X and an isomorphism $\varphi\colon p_1^*(\mathcal{N}) \simeq p_2^*(\mathcal{N})$ that satisfies the compatibility relation*

$$p_{13}^*(\varphi) = p_{23}^*(\varphi) \circ p_{12}^*(\varphi).$$

In the proposition, p_1 and p_2 are the two projections $X \times_Y X \to X$, while the morphisms p_{13}, p_{23}, and p_{12} are the 3 projections $X \times_Y X \times_Y X \to X \times_Y X$. We note that in the setting of the proposition, if we start with a quasi-coherent sheaf $\mathcal{M}$ on Y, then $\mathcal{N} = f^*(\mathcal{M})$ and the isomorphism φ is induced by the fact that both $p_1^*(\mathcal{N})$ and $p_2^*(\mathcal{N})$ are canonically isomorphic to the pull-back of $\mathcal{M}$ by $f \circ p_1 = f \circ p_2$.

2. Basics of étale cohomology

We now give a brief introduction to the étale topology and the cohomology of sheaves in this context. For details and proofs, we refer to [Mil80].

The basic idea behind étale topology is to replace the Zariski topology on an algebraic variety by a finer topology. In fact, this is not a topology in the usual sense, but a *Grothendieck topology*. Sheaf theory, and in particular sheaf cohomology still make sense in this setting, and one can extract invariants, for example, from cohomology with coefficients in a finite Abelian group.

As a motivation, note that in the case of a smooth, projective, complex algebraic variety we would like to recover the singular cohomology, with suitable coefficients. The idea consists in "refining" the Zariski topology, which as it stands, does not reflect the classical topology. The key is the notion of étale morphism. Recall that a morphism of complex algebraic varieties $f\colon X \to Y$ is étale if and only if the corresponding map $f\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ is a local isomorphism.

Let X be a fixed Noetherian scheme. The role of the open subsets of X will be played by the category $\acute{\mathrm{E}}\mathrm{t}(X)$ of étale schemes $Y \to X$ over X. Instead of considering inclusions between open subsets, we consider morphisms in $\acute{\mathrm{E}}\mathrm{t}(X)$ (note that if Y_1 and Y_2 are étale schemes over X, any morphism $Y_1 \to Y_2$ of schemes over X is étale). The category $\acute{\mathrm{E}}\mathrm{t}(X)$ has fiber products. The role of open covers is played by *étale covers*: these are families $(U_i \xrightarrow{f_i} U)_i$ of étale schemes over X such that $U = \bigcup_i f_i(U_i)$. The set of étale covers of U is denoted by $\mathrm{Cov}(U)$.

What makes this data into a Grothendieck topology is the fact that it satisfies the following conditions:

- (C1) If $\varphi\colon U \to V$ is an isomorphism in $\acute{\mathrm{E}}\mathrm{t}(X)$, then $(\varphi) \in \mathrm{Cov}(V)$.
- (C2) If $(U_i \to U)_i \in \mathrm{Cov}(U)$ and for every i we have $(U_{i,j} \to U_i)_j \in \mathrm{Cov}(U_i)$, then $(U_{i,j} \to U)_{i,j} \in \mathrm{Cov}(U)$.
- (C3) If $(U_i \to U)_i \in \mathrm{Cov}(U)$, and $V \to U$ is a morphism in $\acute{\mathrm{E}}\mathrm{t}(X)$, then we have $(U_i \times_U V \to V)_i \in \mathrm{Cov}(V)$.

This Grothendieck topology is the *étale topology* on X.

It follows from definition that if $U \in \acute{\mathrm{E}}\mathrm{t}(X)$, and if $(U_i)_i$ is an open cover of U, then $(U_i \to U)_i$ is in $\mathrm{Cov}(U)$. Another important type of étale cover is the following. A finite étale morphism $V \to U$ is a *Galois cover* with group G if G acts (on the right) on V over

U, and if the natural morphism

$$\bigsqcup_{g\in G} V_g \to V\times_U V,\ y\in V_g\to (y,yg)$$

is an isomorphism, where $V_g = V$ for every $g\in G$. It is a general fact that every finite étale morphism $V\to U$ can be dominated by a Galois cover $W\to U$.

Once we have a Grothendieck topology on X, we have corresponding notions of presheaves and sheaves. An étale presheaf on X (say, of sets, or groups, or Abelian groups) is a contravariant functor from $\acute{\mathrm{E}}\mathrm{t}(X)$ to the corresponding category. Given such a presheaf $\mathcal{F}$, a morphism $V\to U$ in $\acute{\mathrm{E}}\mathrm{t}(X)$, and $s\in\mathcal{F}(U)$, we write $s|_V$ for the image of s in $\mathcal{F}(V)$.

An étale presheaf $\mathcal{F}$ of Abelian groups is a sheaf if for every $U\in\acute{\mathrm{E}}\mathrm{t}(X)$ and every étale cover $(U_i\to U)_i$, the following complex

$$0\to\mathcal{F}(U)\to\prod_i\mathcal{F}(U_i)\to\prod_{i,j}\mathcal{F}(U_i\times_U U_j)$$

is exact (a similar definition applies for sheaves of sets or groups, though in that case the condition should be interpreted as saying that $\mathcal{F}(U)\to\prod_i\mathcal{F}(U_i)$ is the equalizer for the obvious two arrows to $\prod_{i,j}\mathcal{F}(U_i\times_U U_j)$). In particular, $\mathcal{F}$ defines a sheaf $\mathcal{F}_U$ on U, in the usual sense, for every U in $\acute{\mathrm{E}}\mathrm{t}(X)$. On the other hand, if $V\to U$ is a Galois cover in $\acute{\mathrm{E}}\mathrm{t}(X)$ with group G, then the corresponding condition on $\mathcal{F}$ is that $\mathcal{F}(U)\simeq\mathcal{F}(V)^G$ (note that G has a natural action on $\mathcal{F}(V)$ since $\mathcal{F}$ is a presheaf). We now consider some examples of étale sheaves.

Example 2.1. If $\mathcal{M}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules on X (in the usual sense), then for $U\xrightarrow{f}X$ in $\acute{\mathrm{E}}\mathrm{t}(X)$ we put

$$W(\mathcal{M})(U)=\Gamma\big(U,f^*(\mathcal{M})\big).$$

It is a consequence of Proposition 1.1 that $W(\mathcal{M})$ is an étale sheaf on X. Abusing notation, we usually denote $W(\mathcal{M})$ simply by $\mathcal{M}$.

Example 2.2. If A is any abelian group, then we get an étale *constant sheaf* on X that takes every $U\to X$ in $\acute{\mathrm{E}}\mathrm{t}(X)$ to $A^{\pi_0(U)}$, where $\pi_0(U)$ is the set of connected components of U. This is denoted by A_X, but whenever the scheme X is understood, we drop the subscript.

Example 2.3. Suppose that Y is a scheme over X. We can attach to Y an étale presheaf of sets $\mathbf{Y}$ on X by defining for $U\to X$ in $\acute{\mathrm{E}}\mathrm{t}(X)$, $\mathbf{Y}(U)=\mathrm{Hom}_X(U,Y)$. It is a consequence of Proposition 1.2 that this is, in fact, an étale sheaf on X. Note that if Y is an (Abelian) group scheme over X, then $\mathbf{Y}$ is a sheaf of (Abelian) groups. For example, if we consider $G_m = X\times_{\mathrm{Spec}\,\mathbf{Z}}\mathrm{Spec}\,\mathbf{Z}[t,t^{-1}]$, then $\mathbf{G}_m(U)$ is the set $\mathcal{O}(U)^*$ of invertible elements in $\mathcal{O}(U)$. Another example is given by the closed subscheme

$$\mu_n = X\times_{\mathrm{Spec}\,\mathbf{Z}}\mathrm{Spec}\,\mathbf{Z}[t]/(t^n-1)\hookrightarrow G_m.$$

In this case we have $\mu_n(U)=\{u\in\mathcal{O}_X(U)\mid u^n=1\}$. For a non-Abelian example, consider the sheaf $\mathbf{GL}_n$ associated to $X\times_{\mathrm{Spec}\,\mathbf{Z}}\mathrm{GL}_{n,\mathbf{Z}}$. In this case we have $\mathbf{GL}_n(U)=\mathrm{GL}_n\big(\mathcal{O}(U)\big)$.

Example 2.4. As a final example, consider the case when $X = \operatorname{Spec} k$, where k is a field. Note that in this case an object in $\text{Ét}(X)$ is just a disjoint union of finitely many Spec K_i, where the K_i are finite, separable extensions of k. It is clear that every étale sheaf $\mathcal{F}$ over X is determined by its values $M_K := \mathcal{F}(\operatorname{Spec} K)$, for K/k as above. Furthermore, $G(K/k)$ has an induced action on M_K, and for every Galois extension L/K of finite, separable extensions of k, we have a functorial isomorphism $M_K \simeq (M_L)^{G(L/K)}$. Let $M := \varinjlim_{K/k} M_K$. This carries a continuous action of $G = G(k^{\text{sep}}/k)$, where k^{sep} is a separable closure of k (the action being continuous means that the stabilizer of every element in M is an open subgroup of G). One can show that this defines an equivalence of categories between the category of étale sheaves on $\operatorname{Spec} k$ and the category of Abelian groups with a continuous G-action.

Suppose that X is an arbitrary Noetherian scheme. It is easy to see that the category $\mathbf{Psh}_{\text{ét}}(X)$ of étale presheaves of Abelian groups on X is an Abelian category. If $\mathbf{Sh}_{\text{ét}}(X)$ is the category of étale sheaves of Abelian groups on X, then one can show that the natural inclusion $\mathbf{Sh}_{\text{ét}}(X) \hookrightarrow \mathbf{Psh}_{\text{ét}}(X)$ has a left adjoint, that takes an étale presheaf $\mathcal{F}$ to the associated étale sheaf. Using this, it is easy to see that also $\mathbf{Sh}_{\text{ét}}(X)$ is an Abelian category. We note that a complex of étale sheaves of Abelian groups on X

$$\mathcal{F}' \stackrel{u}{\to} \mathcal{F} \stackrel{v}{\to} \mathcal{F}''$$

is exact if and only if for every $U \to X$ in $\text{Ét}(X)$, every $a \in \mathcal{F}(U)$ such that $v(a) = 0$, and every $x \in U$, there is $f\colon V \to U$ in $\text{Ét}(X)$ with $x \in f(V)$, such that $a|_V \in \mathcal{F}(V)$ lies in $\operatorname{Im}\big(\mathcal{F}'(V) \to \mathcal{F}(V)\big)$.

Example 2.5. Suppose now that X is a scheme over a field k, and n is a positive integer, not divisible by $\operatorname{char}(k)$. In this case we have an exact sequence of étale sheaves, the *Kummer sequence*

$$0 \to \mu_\ell \to \mathbf{G}_m \stackrel{\cdot n}{\to} \mathbf{G}_m \to 0.$$

In order to see that the morphism $\mathbf{G}_m \to \mathbf{G}_m$ that takes u to u^n is surjective, it is enough to note that for every k-algebra A, and every $a \in A^*$, the natural morphism $\varphi\colon A \to B = A[t]/(t^n - a)$ induces an étale, surjective morphism $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$, and there is $b = t \in B$, such that $\varphi(a) = b^n$.

Note that if k is separably closed, then it is clear that for every k-algebra A that is an integral domain, the set $\{u \in A \mid u^n = 1\}$ coincides with the set of n^{th} roots of 1 in k. Suppose, for simplicity, that X is an integral scheme. In this case, the choice of a primitive n^{th} root of 1 gives an isomorphism $\mu_n \simeq (\mathbf{Z}/n\mathbf{Z})_X$ of étale sheaves on X.

If $f\colon X \to Y$ is a morphism of Noetherian schemes, then for every $U \to Y$ in $\text{Ét}(Y)$, we have $X \times_Y U \to X$ in $\text{Ét}(X)$. Furthermore, if $(U_i \to U)_i$ is an étale cover of U, then $(X \times_Y U_i \to X \times_Y U)_i$ is an étale cover of $X \times_Y U$. Using this, it is easy to see that we have a functor $f_*\colon \mathbf{Sh}_{\text{ét}}(X) \to \mathbf{Sh}_{\text{ét}}(Y)$, such that $f_*(\mathcal{F})(U) = \mathcal{F}(X \times_Y U)$. This is a left exact functor, and one can show that it has a left adjoint, denoted by f^*. For example, we have $f^*(A_Y) \simeq A_X$.

One can show that the category $\mathbf{Sh}_{\text{ét}}(X)$ has enough injectives. In particular, for every $U \to X$ in $\text{Ét}(X)$ we can consider the right derived functors of the left exact functor $\mathcal{F} \to \mathcal{F}(U)$. These are written as $H^i_{\text{ét}}(U, \mathcal{F})$, for $i \geq 0$.

Remark 2.6. If F is an étale sheaf on X, then it can be considered also as a sheaf with respect to the Zariski topology. Suppose that F is an étale sheaf of Abelian groups. If we denote its cohomology groups with respect to the Zariski topology by $H^i_{\text{Zar}}(X, F)$, then we have canonical morphisms $H^i_{\text{Zar}}(X, F) \to H^i_{\text{ét}}(X, F)$.

Example 2.7. If $\mathcal{M}$ is a quasi-coherent sheaf on X, and $W(\mathcal{M})$ is the corresponding étale sheaf associated to $\mathcal{M}$ as in Example 2.1, then one can show (see [Mil80, Remark 2.16]) that the canonical morphisms $H^i(X, \mathcal{M}) \to H^i_{\text{ét}}\big(X, W(\mathcal{M})\big)$ are isomorphisms.

Example 2.8. Let $X = \operatorname{Spec} k$, where k is a field. If we identify an étale sheaf on X with an abelian group M with a continuous G-action, where $G = G(k^{\text{sep}}/k)$, then the functor of taking global sections for the sheaf gets identified to the functor $M \to M^G$. Therefore its derived functors are given precisely by the Galois cohomology functors.

Example 2.9. One can show that the canonical morphism

$$H^1_{\text{Zar}}(X, \mathbf{G}_m) \to H^1_{\text{ét}}(X, \mathbf{G}_m)$$

is an isomorphism (see Proposition 2.12 below for a more general statement), hence

$$H^1_{\text{ét}}(X, \mathbf{G}_m) \simeq \operatorname{Pic}(X).$$

Suppose now that X is an integral scheme over a separably closed field k, and n is a positive integer that is not divisible by $\operatorname{char}(k)$. It follows from Example 2.5 that we have an exact sequence

$$\Gamma(X, \mathcal{O}_X)^* \xrightarrow{\alpha} \Gamma(X, \mathcal{O}_X)^* \to H^1_{\text{ét}}(X, \mathbf{Z}/n\mathbf{Z}) \to \operatorname{Pic}(X) \xrightarrow{\beta} \operatorname{Pic}(X) \to H^2_{\text{ét}}(X, \mathbf{Z}/n\mathbf{Z}),$$

where both α and β are given by taking the n^{th}-power.

The following result (see [Mil80, Theorem II.3.12]) shows that for smooth complex varieties and for finite Abelian group coefficients, étale cohomology agrees with singular cohomology.

Theorem 2.10. *If X is a smooth complex algebraic variety, then for every finite Abelian group A and every $p \geq 0$, we have a canonical isomorphism*

$$H^p(X, A) \simeq H^p_{\acute{e}t}(X, A),$$

where the group on the left-hand side is the singular cohomology group of X^{an} with coefficients in A.

We also need the computation of étale cohomology via Čech cohomology. We only need this for the first cohomology group, but we need it also for sheaves of non-Abelian groups. We recall here the relevant definitions. Suppose that we have an étale sheaf of groups G on X, written multiplicatively. Given a cover $\mathcal{U} = (U_i \to X)_{i \in I}$, we write $U_{ij} = U_i \times_X U_j$ and $U_{ijk} = U_i \times_X U_j \times_X U_k$. A *1-cocycle* for G with respect to $\mathcal{U}$ is a family $(g_{ij})_{i,j \in I}$ with $g_{ij} \in G(U_{ij})$ such that for every i, j, k we have

$$g_{ij}|_{U_{ijk}} = g_{ik}|_{U_{ijk}} g_{kj}|_{U_{ijk}}.$$

One defines an equivalence relation on such 1-cocycles by saying the two cocycles (g_{ij}) and (h_{ij}) are *cohomologous* if there is a family $(p_i)_{i\in I}$, with $p_i \in G(U_i)$, such that

$$h_{ij} = p_i|_{U_{ij}} g_{ij} (p_j|_{U_{ij}})^{-1} \quad \text{for all} \quad i, j \in I.$$

The quotient by this equivalence relation is denoted $\check{H}^1_{\text{ét}}(\mathcal{U}/X, G)$. Note that if G is a sheaf of Abelian groups, then this quotient is an Abelian group. However, in general it is only a pointed set, that is, a set with a distinguished element given by the 1-cocycles cohomologous to $(1_{G(U_i)})_i$.

By taking the direct limit over all étale covers of X, we obtain the set

$$\check{H}^1_{\text{ét}}(X, G) := \varinjlim_{\mathcal{U}} \check{H}^1_{\text{ét}}(\mathcal{U}/X, G).$$

In general, this is a pointed set. However, when G is a sheaf of Abelian groups, then $\check{H}^1_{\text{ét}}(X, G)$ is an Abelian group and we have a canonical isomorphism

$$\check{H}^1_{\text{ét}}(X, G) \simeq H^1_{\text{ét}}(X, G).$$

The above construction is functorial: given a morphism $G_1 \to G_2$ of étale sheaves of groups on X, we obtain a map of pointed sets

$$\check{H}^1_{\text{ét}}(X, G_1) \to \check{H}^1_{\text{ét}}(X, G_2).$$

Recall that a sequence of sheaves of groups

$$1 \to G' \to G \to G'' \to 1$$

is exact if for every U étale over X, we have that $G'(U)$ is the kernel of $G(U) \to G''(U)$, and for every $\alpha \in G''(U)$ and every $x \in U$, there is an étale morphism $g\colon V \to U$ containing x in its image, and such that $s|_V$ lies in the image of $G(V)$. Given such a sequence, we have an exact sequence of pointed sets[1]

$$1 \to G'(X) \to G(X) \to G''(X) \xrightarrow{\delta} \check{H}^1_{\text{ét}}(X, G') \to \check{H}^1_{\text{ét}}(X, G) \to \check{H}^1_{\text{ét}}(X, G'').$$

Moreover, if G' has the property that $G'(U)$ lies in the center of $G(U)$ for all U étale over X, then there is also a boundary map

$$\check{H}^1_{\text{ét}}(X, G'') \to H^2_{\text{ét}}(X, G')$$

such that the whole sequence of pointed sets remains exact. Indeed, this is easy to define at the level of Čech cohomolgy and when X is quasiprojective over an affine scheme étale cohomology agrees with the corresponding Čech cohomology also for H^2; the argument in general makes use of Giraud's nonabelian cohomology (see the proof of [Mil80, Theorem III.2.5]).

Remark 2.11. Recall that if G is an étale sheaf of groups on X, we may consider it as a sheaf with respect to the Zariski topology. We can then define, as above, a Čech cohomology group $\check{H}^1_{\text{Zar}}(X, G)$ and we clearly have a canonical map of pointed sets

$$\check{H}^1_{\text{Zar}}(X, G) \to \check{H}^1_{\text{ét}}(X, G).$$

[1] The exactness in this case means that given any map in the sequence, the inverse image of the distinguished element is the image of the previous map in the sequence.

In the next section we will make use of the following proposition. Note that the pointed set $\check{H}^1_{\mathrm{Zar}}(X, \mathbf{GL}_n)$ is in bijection with the set of isomorphism classes of rank n vector bundles on X.

Proposition 2.12. *For every positive integer n, the canonical map*

$$\check{H}^1_{Zar}(X, \mathbf{GL}_n) \to \check{H}^1_{\acute{e}t}(X, \mathbf{GL}_n)$$

is a bijection.

Sketch of proof. The key step is to show that $\check{H}^1_{\text{ét}}(X, \mathbf{GL}_n) = 0$ when $X = \mathrm{Spec}(A)$, with A a local ring. Indeed, supposed that we have a 1-cocycle g corresponding to an étale, surjective map $f\colon Y = \mathrm{Spec}(B) \to X$. Therefore $g = (g_{ij}) \in GL_n(B \otimes_A B)$. We can interpret g as an isomorphism $\varphi\colon p_1^*(\mathcal{N}) \to p_2^*(\mathcal{N})$, where p_1 and p_2 are the two projections $Y \times_X Y \to Y$ and $\mathcal{N} = \mathcal{O}_Y^{\oplus n}$. Moreover, the cocycle condition says that the compatibility condition in Remark 1.3 is satisfied, hence $\mathcal{N} \simeq f^*(\mathcal{M})$, for a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$, such that φ is the canonical corresponding isomorphism. Since f is faithfully flat, it follows that $\mathcal{M}$ is locally free, hence free, and it is easy to deduce from this that g is the distinguished element. □

3. The (cohomological) Brauer group

Given a Noetherian scheme X, we define the Brauer group of X to be the what is usually called the cohomological Brauer group, namely

$$\mathrm{Br}(X) := H^2_{\text{ét}}(X, \mathbf{G}_m)_{\text{tors}}.$$

Remark 3.1. If X is a smooth variety over a field k, then in fact $H^2_{\text{ét}}(X, \mathbf{G}_m)$ is torsion (see [Mil80, Proposition 2.15]). However, we will not need this fact.

Our first goal is to relate the Brauer group of X to geometry. This is done via the following key notion. For simplicity, we work over an algebraically closed field k.

Definition 3.2. A flat, proper morphism $f\colon X \to Y$ of varieties over k is a $\mathbf{P}^n$-fibration, for some positive integer n, if for every (closed point) $y \in Y$, the fiber $f^{-1}(y)$ is isomorphic to $\mathbf{P}^n$.

Example 3.3. If $\mathcal{E}$ is a locally free sheaf of rank $(n+1)$ on Y, then $f\colon \mathbf{P}(\mathcal{E}) \to Y$ is a $\mathbf{P}^n$-fibration. We will say that a $\mathbf{P}^n$-fibration $f\colon X \to Y$ is a projective bundle if X is isomorphic over Y to some $\mathbf{P}(\mathcal{E})$. We are interested in the existence of $\mathbf{P}^n$-fibrations that are *not* projective bundles. Note that a sufficient criterion for a $\mathbf{P}^n$-fibration to not be a projective bundle is to not have a rational section.

The next proposition shows that $\mathbf{P}^n$-fibrations are locally projective bundles, where "locally" is understood in the sense of the étale topology.

Proposition 3.4. *If $f\colon X \to Y$ is a $\mathbf{P}^n$-fibration, then for every $y \in Y$, there is an étale map $V \to Y$ whose image contains y, such that $X \times_Y V \to V$ is isomorphic over V to $\mathbf{P}^n_V$.*

Proof. The key fact is that the variety $\mathbf{P}^n$ is *rigid*, in the sense that it has no nontrivial infinitesimal deformations. More generally, this is an assertion that holds for all smooth projective varieties Z such that $H^1(Z, T_Z) = 0$ (see [Ser06, Corollary 1.2.15]). However, we will give an ad-hoc proof in the case of $\mathbf{P}^n$.

Step 1. We show that if $(R, \mathfrak{m}, k)$ is a complete local ring and $X \to \mathrm{Spec}(R)$ is a proper, flat morphism whose special fiber $X_0 = X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k)$ is isomorphic to $\mathbf{P}^n_k$, then X is isomorphic to $\mathbf{P}^n_R$. Consider a non-negative integer s and let

$$X_s = X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\mathfrak{m}^{s+1}).$$

We begin by showing that for every s, we can find a line bundle $\mathcal{L}_s$ on X_s such that the following hold:

i) We have $\mathcal{L}_{s+1}|_{X_s} \simeq \mathcal{L}_s$.
ii) The induced map $H^0(X_{s+1}, \mathcal{L}_{s+1}) \to H^0(X_s, \mathcal{L}_s)$ is surjective.
iii) We have an isomorphism $(X_s, \mathcal{L}_s) \simeq \big(\mathbf{P}^n_{R/\mathfrak{m}^{s+1}}, \mathcal{O}(1)\big)$.

When $s = 0$, the existence of $\mathcal{L}_s$ satisfying iii) follows by hypothesis. We now assume the existence of $\mathcal{L}_{s-1}$ and construct $\mathcal{L}_s$. Since X is flat over R, we have an exact sequence

$$0 \to \mathfrak{m}^s/\mathfrak{m}^{s+1} \otimes_k \mathcal{O}_{X_0} \to \mathcal{O}_{X_s} \to \mathcal{O}_{X_{s-1}} \to 0 \tag{1}$$

which induces another exact sequence

$$0 \to \mathfrak{m}^s/\mathfrak{m}^{s+1} \otimes_k \mathcal{O}_{X_0} \to \mathcal{O}^*_{X_s} \to \mathcal{O}^*_{X_{s-1}} \to 0. \tag{2}$$

Since we have an isomorphism $X_0 \simeq \mathbf{P}^n_k$, we deduce

$$H^1(X_0, \mathcal{O}_{X_0}) = 0 = H^2(X_0, \mathcal{O}_{X_0}).$$

By taking the long exact sequence in cohomology corresponding to (2), we obtain an isomorphism

$$\mathrm{Pic}(X_s) \simeq \mathrm{Pic}(X_{s-1}).$$

In particular, there is $\mathcal{L}_s \in \mathrm{Pic}(X_s)$ such that $\mathcal{L}_s|_{X_{s-1}} \simeq \mathcal{L}_{s-1}$. By tensoring (1) with $\mathcal{L}_s$ and taking global sections, we obtain, using the fact that $\mathcal{L}_s|_{X_0} \simeq \mathcal{O}(1)$, an exact sequence

$$H^0(X_s, \mathcal{L}_s) \to H^0(X_{s-1}, \mathcal{L}_{s-1}) \to \mathfrak{m}^s/\mathfrak{m}^{s+1} \otimes_k H^1\big(\mathbf{P}^n_k, \mathcal{O}(1)\big) = 0.$$

We can thus lift a basis of $H^0(X_{s-1}, \mathcal{L}_{s-1}) \simeq (R/\mathfrak{m}^s)^{\oplus(n+1)}$ to sections in $H^0(X_s, \mathcal{L}_s)$. These clearly generate $\mathcal{L}_s$ and thus define a morphism

$$\varphi_s \colon X_s \to \mathbf{P}^n_{R/\mathfrak{m}^{s+1}}.$$

It is straightforward to see that since X_s is flat over $R/\mathfrak{m}^{s+1}$ and φ_s induces an isomorphism $X_{s-1} \simeq \mathbf{P}^n_{R/\mathfrak{m}^s}$, the morphism φ_s is an isomorphism.

It follows from Grothendieck's existence theorem (see [Gro61, Théorème 5.1.4]) that there is a line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ such that $\mathcal{L}|_{X_s} \simeq \mathcal{L}_s$. Moreover, the theorem on formal functions gives an isomorphism

$$H^0(X, \mathcal{L}) \simeq \varprojlim_s H^0(X_s, \mathcal{L}_s)$$

and we thus obtain sections $s_0, \ldots, s_n \in H^0(X, \mathcal{L})$ that span $\mathcal{L}$. The induced morphism $X \to \mathbf{P}^n_R$ is thus an isomorphism, since this is the case for the induced morphisms $X_s \to \mathbf{P}^n_{R/\mathfrak{m}^{s+1}}$ (see [Gro61, Théorème 5.4.1]). This completes the proof of Step 1.

Step 2. Suppose now that $f \colon X \to Y$ is as a $\mathbf{P}^n$-fibration, and for a closed point $y \in Y$, consider the commutative diagram with Cartesian squares

$$\begin{array}{ccccc} X_2 & \longrightarrow & X_1 & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec}(R_2) & \longrightarrow & \operatorname{Spec}(R_1) & \longrightarrow & Y, \end{array}$$

where $R_2 = \widehat{\mathcal{O}_{Y,y}}$ and $R_1 = \mathcal{O}^h_{Y,y}$ is the henselization of $\mathcal{O}_{Y,y}$. We can take as definition

$$R_1 = \varinjlim_{B,\mathfrak{q}} B_{\mathfrak{q}},$$

where the colimit is over the étale $\mathcal{O}_{Y,y}$-algebras B and over the prime ideals $\mathfrak{q}$ of B lying over the maximal ideal of $\mathcal{O}_{Y,y}$ (see [Mil80, Chapter 1.4]).

As we have seen in Step 1, we have an isomorphism $X_2 \simeq \mathbf{P}^n_{R_2}$. It is a consequence of Artin's approximation theorem (see [Art69]) that in this case $X_1 \simeq \mathbf{P}^n_{R_1}$. We deduce from the above definition of the henselization that there is an étale morphism $V \to Y$ containing y in its image such that $X \times_Y V \simeq \mathbf{P}^n_V$. This completes the proof of the proposition. □

We consider the set of isomorphism classes of $\mathbf{P}^n$-fibrations over Y as a pointed set, with the distinguished element corresponding to the trivial $\mathbf{P}^n$-fibration $Y \times \mathbf{P}^n \to Y$. Proposition 3.4 allows us to give a cohomological description of this set. Before stating the result, recall that we have a group scheme over Y

$$PGL_{n+1} = PGL_{n+1,Y} := PGL_{n+1}(k) \times Y,$$

where $PGL_{n+1}(k) \subseteq \mathbf{P}\big(M_{n+1}(k)^*\big)$ is the principal affine open subset defined by the homogeneous polynomial $\det(a_{i,j})$.

Corollary 3.5. *For every $n \geq 1$, we have a bijection of pointed sets between the set of isomorphism classes of $\mathbf{P}^n$-fibrations over Y and $\check{H}^1_{ét}(Y, \mathbf{PGL}_{n+1})$.*

Sketch of proof. Suppose that $\mathcal{U} = (U_i \to Y)_{i \in I}$ is an étale cover of Y and α is a 1-cocycle for $\mathbf{PGL}_{n+1}$ with respect to this cover, that is $\alpha = (\alpha_{ij})_{i,j \in I}$, where the

$$\alpha_{ij} \colon U_{ij} = U_i \times_Y U_j \to PGL_{n+1}(k)$$

satisfy the cocycle condition. For every i in I, consider $X_i := U_i \times \mathbf{P}^n$, and for every $i, j \in I$, we have an isomorphism

$$X_j \times_{U_j} U_{ij} = U_{ij} \times \mathbf{P}^n \to U_{ij} \times \mathbf{P}^n = X_i \times_{U_i} U_{ij}$$

given by $(x, t) \to (x, \alpha_{ij}(x)t)$. In this case, it is a consequence of faithfully flat descent that there is a morphism $X \to Y$, unique up to a canonical isomorphism, such that $X \times_Y U_i \simeq X_i$. The fact that in this way we get a bijection is a consequence of Proposition 3.4 and of the fact that given any variety V, the automorphisms of $V \times \mathbf{P}^n$ over V are in canonical bijection with the maps $V \to PGL_n(k)$. $\square$

Remark 3.6. Consider now "étale rank n vector bundles" on Y, that is, morphisms $f\colon X \to Y$ such that we have an étale cover $(V_i \to Y)_i$ with $X \times_Y V_i$ isomorphic over V_i to $\mathbf{A}^n \times V_i$, and such that the induced transition maps

$$\mathbf{A}^n \times V_{ij} \to \mathbf{A}^n \times V_{ij}$$

(where $V_{ij} = V_i \times_X V_j$) are given by elements in $GL_n\big(\mathcal{O}(V_{ij})\big)$. Arguing as in the proof of Corollary 3.5, we see that the pointed set of isomorphism classes of étale rank n vector bundles on Y is in bijection with $\check{H}^1_{\text{ét}}(Y, \mathbf{GL}_n)$. We deduce from Proposition 2.12 that this is also in bijection with the pointed set of isomorphism classes of rank n vector bundles on Y.

Proposition 3.7. *If the characteristic of k does not divide $(n+1)$, then to every $\mathbf{P}^n$-fibration $f\colon X \to Y$ we can associate an element $\alpha(f) \in \mathrm{Br}(Y)$ such that $\alpha(f) = 0$ if and only if f is a projective bundle.*

Proof. We have an exact sequence of étale sheaves of groups on Y:

$$0 \to \mathbf{G}_m \to \mathbf{GL}_{n+1} \to \mathbf{PGL}_{n+1} \to 0.$$

In fact, this is exact also in the Zariski topology: for the surjectivity assertion, note that the map $\mathrm{GL}_{n+1}(k) \to \mathrm{PGL}_{n+1}(k)$ is Zariski locally trivial, with fiber k^*.

Note that for every étale map $U \to Y$, the Abelian group $\mathbf{G}_m(U)$ lies in the center of $\mathbf{GL}_{n+1}(U)$. We thus obtain an exact sequence of pointed sets

$$\check{H}^1_{\text{ét}}(Y, \mathbf{G}_m) \to \check{H}^1_{\text{ét}}(Y, \mathbf{GL}_{n+1}) \xrightarrow{\beta} \check{H}^1_{\text{ét}}(Y, \mathbf{PGL}_{n+1}) \xrightarrow{\alpha} \check{H}^1_{\text{ét}}(Y, \mathbf{G}_m).$$

We have seen in Corollary 3.5 that the set of isomorphism classes of $\mathbf{P}^n$-fibrations is in bijection with $\check{H}^1_{\text{ét}}(Y, \mathbf{PGL}_{n+1})$.

If fact, the image of α lies in the torsion subgroup of $\check{H}^1_{\text{ét}}(Y, \mathbf{G}_m)$, hence in the Brauer group. Indeed, note that we have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \mu_{n+1} & \to & \mathbf{SL}_{n+1} & \to & \mathbf{PGL}_{n+1} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ 0 & \to & \mathbf{G}_m & \to & \mathbf{GL}_{n+1} & \to & \mathbf{PGL}_{n+1} & \to & 0, \end{array}$$

in which the vertical maps are induced by inclusions. The surjectivity in the top sequence follows from the fact that the map $\mathrm{SL}_{n+1}(k) \to \mathrm{PGL}_{n+1}(k)$ is surjective and étale: this implies that for every étale map $U \to Y$ and for every morphism $\varphi\colon U \to \mathrm{PGL}_{n+1}(k)$, we have a surjective étale map $V \to U$ and a commutative diagram

$$\begin{array}{ccc} V & \to & SL_{n+1}(k) \\ \downarrow & & \downarrow \\ U & \to & PGL_{n+1}(k). \end{array}$$

Indeed, we may take $V = U \times_{\mathrm{PGL}_{n+1}(k)} \mathrm{SL}_{n+1}(k)$. By comparing the exact sequences of pointed sets corresponding to the above two short exact sequences, we see that

$$\mathrm{Im}(\alpha) \subseteq \mathrm{Im}\big(\check{H}^1_{\text{ét}}(Y, \mu_{n+1}) \to \check{H}^1_{\text{ét}}(Y, \mathbf{G}_m)\big) \subseteq \mathrm{Br}(X).$$

In order to complete the proof of the proposition, it is thus enough to show that for a $\mathbf{P}^n$-fibration $f\colon X \to Y$ the corresponding element in $\check{H}^1_{\text{ét}}(Y, \mathbf{PGL}_{n+1})$ lies in the image of $\check{H}^1_{\text{ét}}(Y, \mathbf{GL}_{n+1})$ if and only if f is a projective bundle. Indeed, we have seen in Remark 3.6 that $\check{H}^1_{\text{ét}}(Y, \mathbf{GL}_{n+1})$ is in bijection with the set of isomorphism classes of rank n vector bundles on Y, and it is easy to see that the map β maps a vector bundle $\mathcal{E}$ to $\mathbf{P}(\mathcal{E})$. This completes the proof of the proposition. □

We now relate, in the case of a smooth complex algebraic variety X, the Brauer group and the singular cohomology of X. For every Abelian group A and every positive integer n, we put $A_n := \{a \in A \mid na = 0\}$.

Proposition 3.8. *For every smooth complex algebraic variety X, we have an exact sequence*

$$0 \to \mathrm{Pic}(X)/n \cdot \mathrm{Pic}(X) \to H^2(X, \mathbf{Z}/n\mathbf{Z}) \to \mathrm{Br}(X)_n \to 0.$$

Proof. Consider the Kummer exact sequence

$$0 \to \mathbf{Z}/n\mathbf{Z} \to \mathbf{G}_m \xrightarrow{n} \mathbf{G}_m \to 0.$$

Recall that since X is smooth, it follows from Theorem 2.10 that we have an isomorphism $H^2(X, \mathbf{Z}/n\mathbf{Z}) \simeq H^2_{\text{ét}}(X, \mathbf{Z}/n\mathbf{Z})$. We thus deduce from the above exact sequence, by taking cohomology, the exact sequence

$$\mathrm{Pic}(X) = H^1_{\text{ét}}(X, \mathbf{G}_m) \xrightarrow{\cdot n} H^1_{\text{ét}}(X, \mathbf{G}_m) \to H^2(X, \mathbf{Z}/n\mathbf{Z}) \to H^2_{\text{ét}}(X, \mathbf{G}_m) \xrightarrow{\cdot n} H^2_{\text{ét}}(X, \mathbf{G}_m).$$

By noting that

$$\mathrm{Ker}\big(H^2_{\text{ét}}(X, \mathbf{G}_m) \xrightarrow{\cdot n} H^2_{\text{ét}}(X, \mathbf{G}_m)\big) = \mathrm{Br}(X)_n,$$

we obtain the exact sequence in the proposition. □

Remark 3.9. The map

$$\mathrm{Pic}(X)/n \cdot \mathrm{Pic}(X) \to H^2(X, \mathbf{Z}/n\mathbf{Z})$$

in Proposition 3.8 takes the class of a line bundle $L \in \mathrm{Pic}(X)$ to the image of $c^1(L)$ via the canonical morphism $H^2(X, \mathbf{Z}) \to H^2(X, \mathbf{Z}/n\mathbf{Z})$.

Proposition 3.10. *For every smooth complex algebraic variety X, we have a surjective homomorphism*

$$\mathrm{Br}(X) \to H^3(X, \mathbf{Z})_{\mathrm{tors}}.$$

This is an isomorphism if the first Chern class map $\mathrm{Pic}(X) \to H^2(X, \mathbf{Z})$ is surjective.

Proof. By taking the long exact sequence in cohomology associated to the exact sequence of constant sheaves on X

$$0 \to \mathbf{Z} \xrightarrow{\cdot n} \mathbf{Z} \to \mathbf{Z}/n\mathbf{Z} \to 0,$$

we see that we have an exact sequence

$$H^2(X, \mathbf{Z}) \to H^2(X, \mathbf{Z}/n\mathbf{Z}) \to H^3(X, \mathbf{Z})_n \to 0.$$

Note that in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}(X)/n\cdot\mathrm{Pic}(X) & \xrightarrow{\alpha} & H^2(X,\mathbf{Z}/n\mathbf{Z}) & \longrightarrow & \mathrm{Br}(X)_n & \longrightarrow & 0 \\ & & \uparrow & & \uparrow{\scriptstyle\beta} & & & & \\ & & \mathrm{Pic}(X) & \xrightarrow{c^1} & H^2(X,\mathbf{Z}) & & & & \end{array}$$

we have $\mathrm{Im}(\alpha) \subseteq \mathrm{Im}(\beta)$, with equality if the Chern class map c^1 is surjective. We thus have a surjective map

$$\mathrm{Coker}(\alpha) = \mathrm{Br}(X)_n \to \mathrm{Coker}(\beta) = H^3(X,\mathbf{Z})_n,$$

which is an isomorphism if c^1 is a surjective map. When we vary n, these maps are compatible. Since by definition $\mathrm{Br}(X)$ is torsion, we thus get a surjective map $\mathrm{Br}(X) \to H^3(X,\mathbf{Z})_{\mathrm{tors}}$, which is an isomorphism if c^1 is a surjective map. □

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