APPENDIX 1: REVIEW OF SINGULAR COHOMOLOGY

In this appendix we begin with a brief review of some basic facts about singular homology and cohomology. For details and proofs, we refer to [Mun84]. We then discuss the Leray-Hirsch theorem and the Thom isomorphism, we review some special features of the cohomology of algebraic varieties, and finally, we carry out some simple computations that we need: the cohomology of a projective space and that of a smooth blow-up.

1. Basic facts about singular homology and cohomology

For every Abelian group A and every non-negative integer p, we have a covariant functor $H_p(-,A)$ and a contravariant functor $H^p(-,A)$ from the category of topological spaces to the category of Abelian groups. This is the p^{th} *singular homology group* (respectively, the p^{th} *singular cohomology group*) with coefficients in A. More generally, we have a covariant functor $H_p(-,-,A)$ and a contravariant functor $H^p(-,-,A)$ from the category of pairs of topological spaces[1] to the category of Abelian groups. This is the p^{th} *relative homology group* (respectively, the p^{th} *relative cohomology group*) with coefficients in A. If $f\colon X\to Y$ is a continuous map, the morphism induced in homology is denoted by f_* and that induced in cohomology is denoted by f^*.

These functors satisfy the following properties:

i) If X consists of one point, then
$$H_0(X,A)\simeq A \quad \text{and} \quad H^0(X,A)\simeq A,$$
and
$$H_p(X,A)=0=H^p(X;A) \quad \text{for} \quad p>0.$$
ii) If $X=\bigsqcup_{i\in I}X_i$, then
$$H_p(X,A)\simeq\bigoplus_{i\in I}H_p(X_i,A) \quad \text{for all} \quad p\geq 0, \text{ and}$$
$$H^p(X,A)\simeq\prod_{i\in I}H^p(X_i,A) \quad \text{for all} \quad p\geq 0.$$
iii) If $f,g\colon X\to Y$ are homotopic continuous maps, then they induce the same maps in homology $H_p(X,A)\to H_p(Y,A)$ and in cohomology $H^p(Y,A)\to H^p(X,A)$.
iv) For every pair (X,Y), we have the following functorial long exact sequences:
$$\ldots\to H_p(Y,A)\to H_p(X,A)\to H_p(X,Y,A)\to H_{p-1}(Y,A)\to\ldots$$
and
$$\ldots\to H^p(X,Y,A)\to H^p(X;A)\to H^p(Y,A)\to H^{p+1}(X,Y,A)\to\ldots.$$

[1] A pair (X,Y) consists of a topological space X and a subspace Y; a morphism in this categoy from (X,Y) to (X',Y') is given by compatible continuous maps $X\to X'$ and $Y\to Y'$.

v) (Excision) If (X, Y) is a pair and U is a subset of X such that $\overline{U}$ is contained in the interior of Y, then the inclusion induces isomorphisms

$$H_p(X \smallsetminus U, Y \smallsetminus U, A) \simeq H_p(X, Y, A) \quad \text{and}$$
$$H^p(X, Y, A) \simeq H^p(X \smallsetminus U, Y \smallsetminus U, A) \quad \text{for all} \quad p \geq 0.$$

In what follows we will mostly be interested in the case when the (co)homology has coefficients in $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, or $\mathbf{C}$. An immediate consequence of the definition of singular homology and cohomology groups is that if $\pi_0(X)$ is the set of path-components of X, then

$$H_0(X, A) \simeq A^{(\pi_0(X))} \quad \text{and} \quad H^0(X, A) \simeq \mathrm{Hom}_{\mathbf{Z}}(A, \mathbf{Z})^{\pi_0(X)}. \tag{1}$$

In particular, we have a canonical morphism of Abelian groups

$$\deg\colon H_0(X, A) \to A,$$

which is an isomorphism if X is path-connected.

A useful tool for computing cohomology is provided by the Mayer-Vietoris sequence. Suppose that U and V are open subsets of X such that $X = U \cup V$. If the following maps are given by inclusions:

$$i_U\colon U \hookrightarrow X,\ j_U\colon U \cap V \hookrightarrow U,\ i_V\colon V \hookrightarrow X,\ j_V\colon U \cap V \hookrightarrow V,$$

then for every Abelian group A, we have a long exact sequence:

$$\ldots \longrightarrow H^p(X, A) \xrightarrow{\alpha} H^p(U, A) \oplus H^p(V, A) \xrightarrow{\beta} H^p(U \cap V, A) \longrightarrow H^{p+1}(X, A) \longrightarrow \ldots,$$

where α and β are given by

$$\alpha(y) = (i_U^*(y), i_V^*(y)) \quad \text{and} \quad \beta(y_1, y_2) = j_U^*(y_1) - j_V^*(y_2).$$

A similar sequence holds for homology.

The following is the Universal Coefficient theorem, which describes the groups $H^p(X, A)$ and $H_p(X, A)$ in terms of the groups $H_p(X, \mathbf{Z})$ (see [Mun84, Theorems 55.1, 56.1].

Theorem 1.1. *For every Abelian group A and for every $p \geq 0$, we have canonical short exact sequences*[2]*:*

$$0 \to \mathrm{Ext}^1_{\mathbf{Z}}(H_{p-1}(X, \mathbf{Z}), A) \to H^p(X, A) \to \mathrm{Hom}_{\mathbf{Z}}(H_p(X, \mathbf{Z}), A) \to 0$$

and

$$0 \to H_p(X, \mathbf{Z}) \otimes_{\mathbf{Z}} A \to H_p(X, A) \to \mathrm{Tor}_1^{\mathbf{Z}}(H_{p-1}(X, \mathbf{Z}), A) \to 0.$$

Both sequences are split, but the splitting is non-canonical.

Remark 1.2. An immediate consequence of the theorem is that if K is a field containing $\mathbf{Q}$, then

$$H_p(X, K) \simeq H_p(X, \mathbf{Z}) \otimes_{\mathbf{Z}} K \quad \text{and} \quad H^p(X, K) \simeq H^p(X, \mathbf{Z}) \otimes_{\mathbf{Z}} K \simeq H_p(X, K)^*.$$

[2] We make here the convention that $H_{-1}(X, \mathbf{Z}) = 0$.

Remark 1.3. Since $H_0(X,\mathbf{Z})$ is a free Abelian group, it is an immediate consequence of Theorem 1.1 that

$$H^1(X,\mathbf{Z})\simeq \mathrm{Hom}_{\mathbf{Z}}\big(H_1(X,\mathbf{Z}),\mathbf{Z}\big).$$

In particular, we see that $H^1(X,\mathbf{Z})$ has no torsion.

Remark 1.4. Note that if M is a finitely generated Abelian group, then $\mathrm{Ext}^1_{\mathbf{Z}}(M,\mathbf{Z})$ is torsion and $\mathrm{Hom}_{\mathbf{Z}}(M,\mathbf{Z})$ has no torsion. We thus deduce from Theorem 1.1 that if all homology groups $H_i(X,\mathbf{Z})$ are finitely generated, then the torsion subgroup $H^p(X,\mathbf{Z})_{\mathrm{tors}}$ of $H^p(X,\mathbf{Z})$ is isomorphic to $\mathrm{Ext}^1_{\mathbf{Z}}(H_{p-1}(X,\mathbf{Z}),\mathbf{Z})$, and

$$H^p(X,\mathbf{Z})/H^p(X,\mathbf{Z})_{\mathrm{tors}}\simeq \mathrm{Hom}_{\mathbf{Z}}(H_p(X,\mathbf{Z}),\mathbf{Z}).$$

If R is a ring, then

$$H^*(X,R):=\bigoplus_{p\geq 0}H^p(X,R)$$

has the structure of a graded-commutative graded ring with respect to the *cup-product*

$$H^p(X,R)\times H^q(X,R)\ni (a,b)\to a\cup b\in H^{p+q}(X,R).$$

We also have a *cap product* map

$$H^p(X,R)\times H_q(X,R)\to H_{q-p}(X,R).$$

This makes $H_*(X,R):=\bigoplus_p H_p(X,R)$ a left module over $H^*(X,R)$. These operations satisfy the *projection formula*: if $f\colon X\to Y$ is a continuous map, then

$$f_*\big(f^*(\alpha)\cap\beta\big)=\alpha\cap f_*(\beta)\quad\text{for every}\quad \alpha\in H^*(Y,R),\beta\in H_*(X,R). \tag{2}$$

One can define the cup-product more generally for relative cohomology: if R is a ring and $Y_1,Y_2\subseteq X$, then we have a cup-product map

$$H^p(X,Y_1,R)\times H^q(X,Y_2,R)\to H^{p+q}(X,Y_1\cup Y_2,R)$$

that extends the one in the absolute case and which satisfies similar properties.

Given two topological spaces X and Y, the Künneth theorem computes the (co)homology of the product $X\times Y$ in terms of the (co)homologies of X and Y. The simplest form concerns homology and says that for every $m\geq 0$, there is a natural short exact sequence

$$0\to\bigoplus_{p+q=m}H_p(X,\mathbf{Z})\otimes_{\mathbf{Z}}H_q(Y,\mathbf{Z})\to H_m(X\times Y,\mathbf{Z})\to\bigoplus_{p+q=m-1}\mathrm{Tor}_1^{\mathbf{Z}}\big(H_p(X,\mathbf{Z}),H_q(Y,\mathbf{Z})\big)\to 0,$$

which is split, but non-canonically. If all homology groups $H_p(X,\mathbf{Z})$ are finitely generated, then for every $m\geq 0$ we have a natural short exact sequence for cohomology:

$$0\to\bigoplus_{p+q=m}H^p(X,\mathbf{Z})\otimes_{\mathbf{Z}}H^q(Y,\mathbf{Z})\xrightarrow{\vartheta}H^m(X\times Y,\mathbf{Z})\to\bigoplus_{p+q=m+1}\mathrm{Tor}_1^{\mathbf{Z}}\big(H^p(X,\mathbf{Z}),H^q(Y,\mathbf{Z})\big)\to 0,$$

where $\vartheta(\alpha\otimes\beta)=\mathrm{pr}_1^*(\alpha)\cup\mathrm{pr}_2^*(\beta)$. Moreover, this sequence splits (non-canonically) if also all $H^q(Y,\mathbf{Z})$ are finitely generated. For these results, see [Mun84, §59, 60].

Suppose now that M is a compact, orientable, n-dimensional $\mathcal{C}^\infty$-manifold. The orientation on M defines on M a *fundamental class* $\mu_M\in H_n(M,\mathbf{Z})$. The following result is one of the incarnations of *Poincaré duality*.

Theorem 1.5. *With the above notation, the cap product map*

$$H^p(M, \mathbf{Z}) \to H_{n-p}(M, \mathbf{Z}), \quad \alpha \to \alpha \cap \mu_M \tag{3}$$

is an isomorphism.

One can show that for a compact manifold M, the groups $H_p(M, \mathbf{Z})$ (hence also the groups $H^p(M, \mathbf{Z})$) are finitely generated. We can now rephrase Poincaré duality as follows. Consider the following bilinear pairing

$$H^p(X, \mathbf{Z}) \times H^{n-p}(X, \mathbf{Z}) \to \mathbf{Z}, \quad (\alpha, \beta) \to \deg\big((\alpha \cup \beta) \cap \mu_M\big). \tag{4}$$

Via the isomorphism (3), this gets identified with the morphism

$$H^p(X, \mathbf{Z}) \times H_p(X, \mathbf{Z}) \to \mathbf{Z}, \quad (\alpha, \beta) \to \deg(\alpha \cap \beta).$$

Using the Universal Coefficient theorem, we see that the map induced after killing the torsion is a perfect pairing. We thus conclude that (4) induces after killing the torsion a perfect pairing:

$$H^p(X, \mathbf{Z})/H^p(X, \mathbf{Z})_{\mathrm{tors}} \times H^{n-p}(X, \mathbf{Z})/H^{n-p}(X, \mathbf{Z})_{\mathrm{tors}} \to \mathbf{Z}. \tag{5}$$

By tensoring with $\mathbf{Q}$, we obtain a perfect pairing

$$H^p(X, \mathbf{Q}) \times H^{n-p}(X, \mathbf{Q}) \to \mathbf{Q}, \quad (\alpha, \beta) \to \deg\big((\alpha \cup \beta) \cap \mu_M\big).$$

2. The Leray-Hirsch theorem and the Thom isomorphism

We now discuss two results that we will need later, in order to compute the cohomology of projective bundle and that of smooth blow-ups. We begin with a result that allows the description of the cohomology for the total space of a locally trivial map. Recall that a continuous map $f\colon Y \to X$ is *locally trivial, with fiber* F, if X can be covered by open subsets U such that $f^{-1}(U)$ is homeomorphic over U with $U \times F$. In the cases of interest to us, X will be covered by finitely many such subsets, and in this case the results below are easier to prove.

Theorem 2.1. *Let $f\colon Y \to X$ be a continuous map, which is locally trivial, with fiber F. Suppose that all cohomology groups $H^p(F, \mathbf{Z})$ are finitely generated, free Abelian groups. If $\alpha_1, \ldots, \alpha_r \in H^*(Y, \mathbf{Z})$ are such that for every $x \in X$, the inclusion $i_x\colon f^{-1}(x) \hookrightarrow Y$ has the property that $i_x^*(\alpha_1), \ldots, i_x^*(\alpha_r)$ give a basis of $H^*(f^{-1}(x), \mathbf{Z})$, then we have an isomorphism of Abelian groups*

$$\mathbf{Z}^r \otimes_{\mathbf{Z}} H^*(X, \mathbf{Z}) \to H^*(Y, \mathbf{Z}), \ \ (m_1, \ldots, m_r) \otimes \beta = \left(\sum_{i=1}^r m_i \alpha_i\right) \cup f^*(\beta).$$

Proof. If $Y = X \times F$, then the assertion is an easy consequence of the Künneth theorem. When X has a finite cover by open subsets U_i such that $f^{-1}(U_i)$ is isomorphic over U_i with $U_i \times F$, one argues by induction on the number of open subsets, using the Mayer-Vietoris sequence. We do not give the proof in the general case, as we will not need it. $\square$

We now turn to the definition of the Thom class and to the statement of the Thom isomorphism theorem. Suppose that $\pi\colon E \to X$ is an oriented, real vector bundle of rank $r \geq 1$, on a topological space X. In particular, for every $x \in X$, the fiber $E_x = \pi^{-1}(x)$ is an r-dimensional, oriented real vector space. We will consider X embedded in E via the 0-section.

Recall that

$$H^r(\mathbf{R}^r, \mathbf{R}^r \smallsetminus \{0\}, \mathbf{Z}) \simeq \mathbf{Z},$$

and choosing a generator is equivalent to the choice of an orientation on $\mathbf{R}^r$. Indeed, it follows from the long exact sequence in cohomology for $(\mathbf{R}^r, \mathbf{R}^r \smallsetminus \{0\})$ and from the fact that $\mathbf{R}^r$ is contractible, while $\mathbf{R}^r \smallsetminus \{0\}$ is homotopically equivalent to the sphere S^{r-1}, that[3]

$$H^r(\mathbf{R}^r, \mathbf{R}^r \smallsetminus \{0\}, \mathbf{Z}) \simeq H^{r-1}(\mathbf{R}^r \smallsetminus \{0\}, \mathbf{Z}) \simeq H^{r-1}(S^{r-1}, \mathbf{Z}) \simeq \mathbf{Z}.$$

Theorem 2.2. *Let $\pi\colon E \to X$ be an oriented, real vector bundle of rank $r \geq 1$, on a topological space X.*

i) *There is a unique $\eta_E \in H^r(E, E \smallsetminus X, \mathbf{Z})$ (the* Thom class *of E) such that for every $x \in X$, the restriction of η_E to $H^r(\pi^{-1}(x), \pi^{-1}(x) \smallsetminus \{0\}, \mathbf{Z})$ corresponds to the given orientation on the fiber.*
ii) *For every closed subset W of X and every $p \geq 0$, we have*

$$H^p(E, E \smallsetminus W, \mathbf{Z}) = 0 \quad \text{for} \quad p < r$$

and the map

$$H^{p-r}(X, X \smallsetminus W, \mathbf{Z}) \to H^p(E, E \smallsetminus W, \mathbf{Z}), \quad \alpha \to \pi^*(\alpha) \cup \eta_E$$

is an isomorphism for every $p \geq r$ (the Thom isomorphism*).*

Proof. We only sketch the argument under the assumption that there are finitely many open subsets of X that cover X and such that E is trivial over each of them (this is enough for our purpose). For both i) and ii), arguing by induction on the number of open subsets and using the Mayer-Vietoris sequence, we reduce to the case when E is trivial. In this case the assertions follow easily from the Künneth theorem. □

Remark 2.3. We will apply the Thom isomorphism via the following consequence. Suppose that X is an oriented, n-dimensional $\mathcal{C}^\infty$-manifold, and Y is a closed, oriented submanifold, of codimension r. The orientations on the tangent bundles T_X and T_Y induce an orientation on the normal bundle $N = N_{Y/X}$, since we have a canonical isomorphism $\det(T_X) \simeq \det(T_Y) \otimes \det(N)$. Recall that by the tubular neighborhood theorem, there is an open neighborhood U of Y in X, a retract $r\colon U \to Y$, and a homeomorphism $h\colon U \to N$, such that we have a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "j"] \arrow[d, "1_Y"'] & U \arrow[r, "r"] \arrow[d, "h"'] & Y \arrow[d, "1_Y"'] \\ Y \arrow[r, "i"] & N \arrow[r, "\pi"] & Y, \end{tikzcd}$$

[3]In order for this to also hold for $r = 1$, we need to replace the 0^{th} cohomology by the reduced version, the cokernel of the canonical map $\mathbf{Z} \to H^0(-, \mathbf{Z})$.

where j is the inclusion, i is the embedding as the 0-section, and π is the vector bundle map. If W a a closed subset of Y, we then obtain isomorphisms

$$H^{p-r}(Y, Y \smallsetminus W, \mathbf{Z}) \simeq H^p(N, N \smallsetminus W, \mathbf{Z}) \simeq H^p(U, U \smallsetminus W, \mathbf{Z}) \simeq H^p(X, X, \smallsetminus W, \mathbf{Z}),$$

for every $p \geq 0$, where the first isomorphism is provided by Theorem 2.2, the second one is induced by h, and the third one is given by excision (we make the convention that the first group is 0 for $p < r$). In particular, we have isomorphisms

$$H^{p-r}(Y, \mathbf{Z}) \simeq H^p(X, X \smallsetminus Y, \mathbf{Z}) \quad \text{for all} \quad p \geq 0. \tag{6}$$

Remark 2.4. If X and Y are as in the previous remark and $\iota \colon Y \to X$ is the inclusion, then the Gysin map

$$\iota_* \colon H^p(Y, \mathbf{Z}) \to H^{p+r}(X, \mathbf{Z})$$

is defined as the composition

$$H^p(Y, \mathbf{Z}) \to H^{p+r}(X, X \smallsetminus Y, \mathbf{Z}) \to H^{p+r}(X, \mathbf{Z}),$$

where the first map is the isomorphism in (6) and the second one is the canonical map induced by $(X, \emptyset) \to (X, X \smallsetminus Y)$. Therefore the long exact sequence in cohomology for the pair $(X, X \smallsetminus Y)$ becomes

$$\ldots \to H^{p-r}(Y, \mathbf{Z}) \xrightarrow{\iota_*} H^p(X, \mathbf{Z}) \to H^p(X \smallsetminus Y, \mathbf{Z}) \to H^{p-r+1}(Y, \mathbf{Z}) \to \ldots. \tag{7}$$

One can show that Gysin maps are functorial for closed embeddings of oriented $\mathcal{C}^\infty$-manifolds.

Remark 2.5. If $f \colon Y \to X$ is any differentiable map between compact, oriented $\mathcal{C}^\infty$-manifolds, with $\dim(X) = n$ and $\dim(Y) = m$, we get a morphism

$$f_* \colon H^p(Y, \mathbf{Z}) \to H^{p+r}(X, \mathbf{Z}),$$

where $r = n - m$. Indeed, Poincaré duality gives isomorphisms

$$H^p(Y, \mathbf{Z}) \simeq H_{m-p}(Y, \mathbf{Z}) \quad \text{and} \quad H^{p+r}(X, \mathbf{Z}) \simeq H_{m-p}(X, \mathbf{Z})$$

and via these isomorphisms, f_* corresponds to the morphism in homology associated to f. It is clear that this is functorial. Moreover, one can show that if f embeds Y as a submanifold of X, then this definition coincides with the one in Remark 2.4.

3. Cohomology of complex algebraic varieties

In this section we review some basic facts about the topology of complex algebraic varieties. For proofs and details, we refer to [Voi07]. We make use of the singular cohomology for complex algebraic varieties. Given such a variety X (not necessarily irreducible), we denote by X^{an} the same variety, endowed with the classical topology. Recall how this is defined: if X is affine, then we have a closed immersion $X \hookrightarrow \mathbf{C}^n$, and we endow X with the subspace topology, where we take on $\mathbf{C}^n$ the Euclidean topology. It is straightforward to see that the topology we get is independent of the embedding. If X is not-necessarily-affine, then we obtain the topology on X^{an} by gluing the topology that we get on the subsets in an affine open cover. In this way, we get a functor from complex algebraic varieties to topological spaces. Whenever we consider the singular cohomology of an algebraic variety over $\mathbf{C}$, we always consider it endowed with the classical topology. The following are some basic facts that we will use:

i) If X is irreducible (or, more generally, connected), then $X^{\rm an}$ is connected.
ii) Since X is assumed to be separated, then $X^{\rm an}$ is Hausdorff.
iii) If X is a complete variety, then $X^{\rm an}$ is compact.
iv) If X is smooth of dimension n, then $X^{\rm an}$ is a complex manifold of dimension n, and therefore also a (real) $\mathcal{C}^\infty$-manifold of dimension $2n$. Like every complex manifold, it carries a canonical orientation.

We note that for every complex algebraic variety X and every Abelian group A, we have canonical isomorphisms

$$H^p(X^{\rm an}, \underline{A}) \simeq H^p(X^{\rm an}, A) \quad \text{for all} \quad p \geq 0,$$

where on the left-hand side we have sheaf cohomology with values in the constant sheaf $\underline{A}$, and on the right-hand side we have the singular cohomology groups with coefficients in A. In fact, such isomorphisms hold for all "nice" topological spaces (see [God73]). From now on, we always identify these two groups. In order to simplify the notation, we write $H_i(X, \mathbf{Z})$ and $H^i(X, \mathbf{Z})$ instead of $H_i(X^{\rm an}, \mathbf{Z})$ and $H^i(X^{\rm an}, \mathbf{Z})$, and similarly for the other singular (co)homology groups.

If Y is a complete complex algebraic variety of dimension n, then $Y^{\rm an}$ admits a finite triangulation, with simplices of dimension $\leq 2n$. More generally, if Z is a closed subvariety of Y, then $Z^{\rm an}$ and $Y^{\rm an}$ admit compatible finite triangulations, with simplices of dimension $\leq 2n$. Given an arbitrary complex algebraic variety X, of dimension n, it can be embedded as an open subvariety of a complete variety Y by Nagata's theorem. Using the above-mentioned fact about triangulations, one can show that the groups $H_i(X, \mathbf{Z})$ and $H^i(X, \mathbf{Z})$ are finitely generated, and they vanish for $i > 2n$. The *Betti numbers* of X are the ranks of these (co)homology groups:

$$b_i(X) := \operatorname{rk} H^i(X, \mathbf{Z}) = \operatorname{rk} H_i(X, \mathbf{Z}).$$

Let X be a complex algebraic variety. The topological space $X^{\rm an}$ carries a sheaf of *holomorphic functions* $\mathcal{O}_{X^{\rm an}}$. In order to define this sheaf, suppose first that X is affine and consider an embedding $X \hookrightarrow \mathbf{C}^n$ as a Zariski closed subset. By definition, holomorphic functions on open subsets of $X^{\rm an}$ are the functions that can be locally extended to holomorphic functions on open subsets of $\mathbf{C}^n$. The definition is independent of embedding and in the case of an arbitrary variety X, the sheaf $\mathcal{O}_{X^{\rm an}}$ is obtained by glueing the corresponding sheaves on the open subsets in an affine open cover. Note that we have a morphism of locally ringed spaces $\iota\colon (X^{\rm an}, \mathcal{O}_{X^{\rm an}}) \to (X, \mathcal{O}_X)$ whose corresponding morphism of topological spaces $X^{\rm an} \to X$ is the identity map.

If $\mathcal{F}$ is a coherent sheaf on X, then one gets a sheaf $\mathcal{F}^{\rm an} := \iota^*\mathcal{F}$ on $X^{\rm an}$ and in this way we get a functor from coherent sheaves on X to coherent sheaves of $\mathcal{O}_{X^{\rm an}}$-modules. Serre's GAGA theorem says that if X is projective, then this functor is an equivalence of categories, preserving the locally free sheaves in each category. Moreover, for every coherent sheaf on X, we have a canonical isomorphism

$$H^p(X, \mathcal{F}) \simeq H^p(X^{\rm an}, \mathcal{F}^{\rm an}).$$

We note that as in the algebraic category, the group $\mathrm{Pic}(X^{\mathrm{an}})$ of rank 1 locally free $\mathcal{O}_{X^{\mathrm{an}}}$-modules is isomorphic to $H^1(X^{\mathrm{an}}, \mathcal{O}^*_{X^{\mathrm{an}}})$. For all these results, we refer to [Ser56].

A very useful tool in this analytic setting is provided by the *exponential sequence*. If X is any complex algebraic variety, then on X^{an} we have a short exact sequence

$$0 \longrightarrow \mathbf{Z}_X \longrightarrow \mathcal{O}_{X^{\mathrm{an}}} \overset{\exp(2\pi i -)}{\longrightarrow} \mathcal{O}^*_{X^{\mathrm{an}}} \longrightarrow 0,$$

where the first map is the inclusion and the second map takes a holomorphisc function φ to $\exp(2\pi i\varphi)$. Exactness is an immediate consequence of the local existence of the logarithm function.

By taking the long exact sequence in cohomology associated to the exponential sequence, we obtain an exact sequence

$$H^1(X, \mathbf{Z}) \longrightarrow H^1(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \longrightarrow H^1(X^{\mathrm{an}}, \mathcal{O}^*_{X^{\mathrm{an}}}) \overset{c^1}{\longrightarrow} H^2(X, \mathbf{Z}).$$

We will also denote by c^1 the composition

$$\mathrm{Pic}(X) \to \mathrm{Pic}(X^{\mathrm{an}}) = H^1(X^{\mathrm{an}}, \mathcal{O}^*_{X^{\mathrm{an}}}) \to H^2(X, \mathbf{Z}).$$

This is the *first Chern class map* for line bundles.

Chern classes of line bundles behave functorially. Indeed, if $f\colon Y \to X$ is a morphism, then we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \underline{\mathbf{Z}} \arrow[r] \arrow[d] & (f^{\mathrm{an}})^{-1}(\mathcal{O}_{X^{\mathrm{an}}}) \arrow[r] \arrow[d] & (f^{\mathrm{an}})^{-1}(\mathcal{O}^*_{X^{\mathrm{an}}}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underline{\mathbf{Z}} \arrow[r] & \mathcal{O}_{Y^{\mathrm{an}}} \arrow[r] & \mathcal{O}^*_{Y^{\mathrm{an}}} \arrow[r] & 0,
\end{tikzcd}$$

and by comparing the corresponding long exact sequences in cohomology, we see that for every $L \in \mathrm{Pic}(X)$, we have $c^1\big(f^*(L)\big) = f^*\big(c^1(L)\big)$.

If X is any complete complex algebraic variety of dimension n, then X carries a *fundamental class* $\mu_X \in H_{2n}(X, \mathbf{Z})$, defined as follows. If X is smooth, then X^{an} is a compact oriented $\mathcal{C}^\infty$-manifold of dimension $2n$ and μ_X is the corresponding fundamental class. Suppose now that X is an arbitrary complete complex n-dimensional variety. By Hironaka's theorem, we have a proper morphism $f\colon Y \to X$ such that Y is smooth, and we put $\mu_X = f_*(\mu_Y) \in H_{2n}(X, \mathbf{Z})$. One can show that this is independent of the choice of f. Moreover, if $g\colon W \to X$ is an arbitrary surjective, generically finite morphism of complete complex algebraic varieties, then

$$f_*(\mu_W) = \deg(f)\mu_X. \tag{8}$$

On the other hand, if g is such that $\dim\big(g(W)\big) < \dim(W)$, then $g_*(\mu_W) = 0$ (this is due to the fact that g_* factors through $H_*\big(g(W), \mathbf{Z}\big)$ and $H_i\big(g(W), \mathbf{Z}\big) = 0$ for $i > 2\cdot\dim\big(g(W)\big)$.

Recall that if $f\colon W \to X$ is a surjective morphism of smooth, complete complex algebraic varieties, with $\dim(W) = \dim(X) = n$, then for every p we have a morphism

$$f_*\colon H^p(W, \mathbf{Z}) \to H^p(X, \mathbf{Z})$$

induced via Poincaré duality by the push-forward in homology (see Remark 2.5). It is easy to deduce from (8) and the projection formula (2) that $f_* \circ f^* = \deg(f) \cdot \mathrm{Id}$. In particular, f^* is injective if f is birational.

Suppose now that X is a complete complex algebraic variety, of dimension n. We have a symmetric, multilinear map

$$H^2(X, \mathbf{Z})^{\times n} \to \mathbf{Z}, (\alpha_1, \ldots, \alpha_n) \to (\alpha_1 \cdot \ldots \cdot \alpha_n) := \deg\big((\alpha_1 \cup \ldots \cup \alpha_n) \cap \mu_X\big).$$

This is compatible with intersection numbers of line bundles via the Chern class map: if $L_1, \ldots, L_n \in \mathrm{Pic}(X)$, then

$$(L_1 \cdot \ldots \cdot L_n) = \big(c^1(L_1) \cdot \ldots \cdot c^1(L_n)\big).$$

Example 3.1. If X is a smooth, n-dimensional, complete complex variety, Y is a smooth, Cartier divisor on X, and $i \colon Y \hookrightarrow X$ is the inclusion, then

$$i_*(\mu_Y) = c^1\big(\mathcal{O}_X(Y)\big) \cap \mu_X \text{ in } H_{2n-2}(X, \mathbf{Z}).$$

Via the projection formula, this implies that for every $\alpha \in H^p(X, \mathbf{Z})$, we have

$$i_*(i^*(\alpha) \cap \mu_Y) = \alpha \cup c^1\big(\mathcal{O}_X(Y)\big) \in H^{p+2}(X, \mathbf{Z}),$$

where we identify cohomology and homology via Poincaré duality.

More generally, suppose that $Y_1, \ldots, Y_r$ are smooth, effective divisors on X that intersect transversely, so that $Y = Y_1 \cap \ldots \cap Y_r$ is smooth, of codimension r in X. An easy induction on n implies that if $i \colon Y \hookrightarrow X$ is the inclusion, then

$$i_*(\mu_Y) = \big(c^1(\mathcal{O}_X(Y_1)) \cup \ldots c^1(\mathcal{O}_X(Y_r))\big) \cap \mu_X.$$

For example, if $X = \mathbf{P}^n$, then for every $d \leq n$, we have $c^1\big(\mathcal{O}_{\mathbf{P}^n}(1)\big)^d \cap \mu_{\mathbf{P}^n} = i_*(\mu_L)$, where $L \subseteq \mathbf{P}^n$ is any linear subspace of codimension d and $i \colon L \hookrightarrow \mathbf{P}^n$ is the inclusion.

We now recall the Hodge decomposition. Suppose that X is a smooth, projective complex algebraic variety. In this case, we have for every $m \geq 0$ a decomposition

$$H^m(X, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{C} = H^m(X, \mathbf{C}) = \bigoplus_{p+q=m} H^{p,q}, \quad \text{where} \quad H^{p,q} \simeq H^q(X, \Omega_X^p).$$

Moreover, the complex conjugation induces a real linear map on $H^m(X; \mathbf{C})$ that maps $H^{p,q}$ to $H^{q,p}$. In particular, we have

$$h^q(X, \Omega_X^p) = h^p(X, \Omega_X^q) \quad \text{for all} \quad p, q \geq 0.$$

Remark 3.2. If X is a smooth, projective complex algebraic variety, then Poincaré duality is compatible with the Hodge decomposition, inducing an isomorphism between $H^{p,q}$ and the dual of $H^{n-p,n-q}$, where $n = \dim(X)$. Suppose now that $f \colon X \to Y$ is a morphism of such varieties. The pull-back $f^* \colon H^*(Y, \mathbf{C}) \to H^*(X, \mathbf{C})$ preserves the Hodge decomposition: for every p and q, we have

$$f^*\big(H^{p,q}(Y)\big) \subseteq H^{p,q}(X),$$

with the map being induced by the morphism $f^*\Omega_Y^p \to \Omega_X^p$. Using the definition of $f_*\colon H^*(X,\mathbf{C}) \to H^*(Y,\mathbf{C})$ and the above compatibility between Poincaré duality and the Hodge decomposition, we deduce that

$$f_*\big(H^{p,q}(X)\big) \subseteq H^{m-n+p,m-n+q}(Y),$$

where $n = \dim(X)$ and $m = \dim(Y)$.

Finally, we need Noether's formula for smooth complex projective surfaces. This says that if X is such a surface, then

$$\chi(X,\mathcal{O}_X) = \frac{1}{12}\big((K_X^2) + \chi_{\rm top}(X)\big),$$

where

$$\chi_{\rm top}(X) = b_0(X) - b_1(X) + b_2(X) - b_3(X) + b_4(X) = 2 - 2q(X) + b_2(X)$$

is the topological Euler-Poincaré characteristic. This formula is a consequence of the Hirzebruch-Riemann-Roch theorem for surfaces (see [Ful98, Example 15.2.2]).

4. Cohomology of projective bundles and smooth blow-ups

We begin with the computation of the cohomology of the complex projective space.

Proposition 4.1. *For every $n \geq 0$, we have a ring isomorphism*

$$H^*(\mathbf{P}^n,\mathbf{Z}) \simeq \mathbf{Z}[x]/(x^{n+1}),$$

mapping $c^1\big(\mathcal{O}_{\mathbf{P}^n}(1)\big)$ to the class of x. In particular, we have $H^i(\mathbf{P}^n,\mathbf{Z}) = 0$ if i is odd or $i > 2n$, and $H^i(\mathbf{P}^n,\mathbf{Z}) \simeq \mathbf{Z}$, otherwise.

Proof. We proceed by induction of n, following the argument in [Voi07, Theorem 7.14]. The assertion is trivial for $n = 0$. In order to prove the induction step, note that we have a closed subvariety Y of $\mathbf{P}^n$ such that $Y \simeq \mathbf{P}^{n-1}$ and $\mathbf{P}^n \smallsetminus Y \simeq \mathbf{C}^n$. The long exact sequence (7) becomes

$$\ldots \to H^{i-2}(\mathbf{P}^{n-1},\mathbf{Z}) \to H^i(\mathbf{P}^n,\mathbf{Z}) \to H^i(\mathbf{C}^n,\mathbf{Z}) \to H^{i-1}(\mathbf{P}^{n-1},\mathbf{Z}) \to \ldots.$$

Since $\mathbf{C}^n$ is contractible, we have $H^i(\mathbf{C}^n,\mathbf{Z}) = 0$ for all $i > 0$, and we conclude from the exact sequence that

$$H^i(\mathbf{P}^n,\mathbf{Z}) \simeq H^{i-2}(\mathbf{P}^{n-1},\mathbf{Z}) \quad \text{for all} \quad i \geq 1$$

and $H^0(\mathbf{P}^n,\mathbf{Z}) \simeq H^0(\mathbf{C}^n,\mathbf{Z}) \simeq \mathbf{Z}$. The last assertion in the proposition thus follows from the inductive assumption.

In order to complete the proof of the induction step, we need to show that if $h = c^1\big(\mathcal{O}_{\mathbf{P}^n}(1)\big)$, then h^k is a generator of $H^{2k}(\mathbf{P}^n,\mathbf{Z})$ for $0 \leq k \leq n$. The key point is that $(h^n) = (\mathcal{O}_{\mathbf{P}^n}(1)^n) = 1$. In particular, we have $h^k \neq 0$ for $0 \leq k \leq n$. For every element $\alpha \in H^{2k}(\mathbf{P}^n,\mathbf{Z})$, we can thus write $\alpha = qh^k$, for some $q \in \mathbf{Q}$. On the other hand, since

$$q = q(h^n) = \deg\big((\alpha \cup h^{n-k}) \cap \eta_{\mathbf{P}^n}\big) \in \mathbf{Z},$$

we conclude that $q \in \mathbf{Z}$. This implies that $H^{2k}(\mathbf{P}^n,\mathbf{Z})$ is generated by α^k, completing the proof. $\square$

Corollary 4.2. *If X is a complex variety and $\pi\colon \mathbf{P}(E)\to X$ is a projective bundle on X, with $\mathrm{rk}(E)=n+1$, then we have a ring isomorphism*

$$H^*(X,\mathbf{Z})[x]/(x^{n+1})\to H^*\big(\mathbf{P}(E),\mathbf{Z}\big),\ \ \sum_{i=0}^{n}\alpha_i x^i\to\sum_{i=0}^{n}\pi^*(\alpha_i)\cup c^1\big(\mathcal{O}_{\mathbf{P}(E)}(1)\big)^i.$$

Proof. It is clear that the map is a ring homomorphism. In order to see that it is an isomorphism, it is enough to apply the Leray-Hirsch theorem for the elements $c^1\big(\mathcal{O}_{\mathbf{P}^n}(1)\big)^i\in H^{2i}(\mathbf{P}(E),\mathbf{Z})$, for $0\leq i\leq n$; the fact that they satisfy the hypothesis in the theorem follows from Proposition 4.1. $\square$

We can now compute the cohomology of a smooth blow-up. Let us first fix some notation. We consider a smooth complex variety X and a smooth closed subvariety Y of X, of codimension r. Let $f\colon \widetilde{X}\to X$ be the blow-up of X along Y, with exceptional divisor E. Let $j\colon E\hookrightarrow\widetilde{X}$ be the inclusion and $g\colon E\to Y$ the map induced by f.

Proposition 4.3. *With the above notation, for every $i\geq 0$, we have an isomorphism*

$$H^p(X,\mathbf{Z})\oplus\bigoplus_{q=1}^{r-1}H^{p-2q}(Y,\mathbf{Z})\simeq H^p(\widetilde{X},\mathbf{Z})\quad\textit{given by}$$

$$(\beta,\alpha_1,\ldots,\alpha_{r-1})\to f^*(\beta)+\sum_{q=1}^{r-1}j_*\big(c^1(g^*(\alpha_q)\cup\mathcal{O}_E(1))^q\big),$$

with the convention that $H^{p-2q}(Y,\mathbf{Z})=0$ if $p-2q<0$.

Proof. Since E is a projective bundle over Y, with fibers isomorphic to $\mathbf{P}^{r-1}$, it follows from Corollary 4.2 that we have an isomorphism

$$\bigoplus_{q=1}^{r}H^{p-2q}(Y,\mathbf{Z})\to H^{p-2}(E,\mathbf{Z}),\ \ (\alpha_1,\ldots,\alpha_r)\to\sum_{q=1}^{r}g^*(\alpha_q)\cup c^1\big(\mathcal{O}_E(1)\big)^{q-1}.$$

By considering the long exact cohomology sequences for the two pairs $(X,X\smallsetminus Y)$ and $(\widetilde{X},\widetilde{X}\smallsetminus E)$, we obtain the following commutative diagram with exact rows:

$$\begin{tikzcd}
H^p(X,X\smallsetminus Y,\mathbf{Z}) \arrow[r] \arrow[d, "f^*"] & H^p(X,\mathbf{Z}) \arrow[r] \arrow[d, "f^*"] & H^p(X\smallsetminus Y,\mathbf{Z}) \arrow[r] \arrow[d] & H^{p+1}(X,X\smallsetminus Y,\mathbf{Z}) \arrow[d, "f^*"] \\
H^p(\widetilde{X},\widetilde{X}\smallsetminus E,\mathbf{Z}) \arrow[r] & H^p(\widetilde{X},\mathbf{Z}) \arrow[r] & H^p(\widetilde{X}\smallsetminus E,\mathbf{Z}) \arrow[r] & H^{p+1}(\widetilde{X},\widetilde{X}\smallsetminus E,\mathbf{Z}),
\end{tikzcd}$$

in which the vertical maps are induced by the pull-back in cohomology corresponding to f. Note that the third vertical map is an isomorphism. Moreover, by using Thom isomorphisms as in Remark 2.4, we obtain a commutative diagram

$$\begin{tikzcd}
H^p(X,X\smallsetminus Y,\mathbf{Z}) \arrow[r] \arrow[d, "f^*"] & H^{p-2r}(Y,\mathbf{Z}) \arrow[d, "\varphi_p"] \\
H^p(\widetilde{X},\widetilde{X}\smallsetminus E,\mathbf{Z}) \arrow[r] & H^{p-2}(E,\mathbf{Z}),
\end{tikzcd}$$

in which the horizontal maps are isomorphisms. Moreover, one can check that

$$\varphi_p(\alpha) = g^*(\alpha) \cup c^1\big(\mathcal{O}_E(1)\big)^{r-1} \quad \text{for al} \quad \alpha \in H^{p-2r}(Y, \mathbf{Z}).$$

Using also the fact that $f^*\colon H^p(X, \mathbf{Z}) \to H^p(\widetilde{X}, \mathbf{Z})$ is injective[4], the assertion in the proposition follows from the above exact sequences, using a straightforward diagram chase. □

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[4] When X and Y are complete, we have seen that this follows from the fact that $f_* \circ f^* = \mathrm{Id}$, and this is the only case that we need. The assertion in the general case follows similarly using Borel-Moore homology and Poincaré duality for non-compact manifolds.