Math 420

Solutions for problems on Midterm 2

(1). (25 points) Let V be a finite-dimensional vector space over a field F and U a linear subspace of V. Recall that V^* is the dual of V and

$$\mathrm{Ann}(U) = \{\varphi \in V^* \mid \varphi(u) = 0 \text{ for all } u \in U\}.$$

Show that if $\pi\colon V \to V/U$ is the map defined by $\pi(v) = v + U$, then π^* induces an isomorphism of $(V/U)^*$ with $\mathrm{Ann}(U)$.

Solution. We have shown in class that for a linear map f, we have f surjective if and only if f^* is injective. Since π is surjective, we conclude that π^* is injective. In fact, this implication is easy, so let's explain the argument in detail: it is enough to show that $\mathrm{null}(\pi^*) = \{0\}$. Suppose that $\varphi \in (V/U)^*$ is such that $\pi^*(\varphi) = 0$. This means that the linear map $\varphi \circ \pi = 0$. Given any element w in V/U, this can be written as $v + U = \pi(v)$ for some $v \in V$. Therefore we have $\varphi(w) = \varphi\big(\pi(v)\big) = 0$, hence $\varphi(w) = 0$ for every $w \in V/U$. This proves that $\varphi = 0$.

In order to complete the proof of the problem, it is enough to show that $\mathrm{range}(\pi^*) = \mathrm{Ann}(U)$. Let's check first that $\mathrm{range}(\pi^*) \subseteq \mathrm{Ann}(U)$. Suppose that $\psi \in V^*$ lies in $\mathrm{range}(\pi^*)$, hence can be written as $\pi^*(\varphi) = \varphi \circ \pi$ for some linear function $\varphi\colon V/U \to F$. In order to show that $\psi \in \mathrm{Ann}(\pi^*)$, we need to show that $\psi(u) = 0$ for every $u \in U$. However, for every such u we have $\pi(u) = 0$, hence $\psi(u) = \varphi\big(\pi(u)\big) = \varphi(0) = 0$. This proves the inclusion $\mathrm{range}(\pi^*) \subseteq \mathrm{Ann}(U)$.

In order to show that in fact we have equality, it is enough to prove that these two vector spaces have the same dimension. One possibility would be to use the following fact that we proved in class:

$$\dim\big(\mathrm{Ann}(U)\big) = \dim(V) - \dim(U).$$

We have already shown that π^* is injective, hence

$$\dim\big(\mathrm{range}(\pi^*)\big) = \dim\big((V/U)^*\big) = \dim(V/U),$$

where the second equality follows from the fact that the dual of a vector space has the same dimension as that vector space. Finally, we showed in class that

$$\dim(V/U) = \dim(V) - \dim(U).$$

We thus conclude that

$$\dim\big(\mathrm{range}(\pi^*)\big) = \dim\big(\mathrm{Ann}(U)\big).$$

Since $\mathrm{range}(\pi^*) \subseteq \mathrm{Ann}(U)$, we see that this is an equality.

Alternatively, we can show directly that every $\psi \in \mathrm{Ann}(U)$ lies in $\mathrm{range}(\pi^*)$. Indeed, $\psi\colon V \to F$ has the property that $\psi(u) = 0$ for every $u \in U$. We define $\varphi\colon V/U \to F$ by $\varphi(v + U) = \psi(v)$. This is well-defined: if $v_1 + U = v_2 + U$, we have $v_1 - v_2 \in U$, hence by assumption $\psi(v_1 - v_2) = 0$. Therefore $\psi(v_1) = \psi(v_2)$. Therefore φ is well-defined. It is clear that φ is linear, hence it gives an element in $(V/U)^*$. Moreover, it follows from the definition that $\varphi \circ \pi = \psi$, hence $\pi^*(\varphi) = \psi$. Therefore $\psi \in \text{range}(\pi^*)$.

(2). (25 points) Let V be a finite-dimensional vector space over a field F. Show that if $T \in \mathcal{L}(V)$, then 9 is an eigenvalue of T^2 if and only if 3 is an eigenvalue of T or -3 is an eigenvalue of T.

Solution. If T has an eigenvalue λ, then there is a nonzero vector $v \in V$ such that $Tv = \lambda v$. Since T is a linear map, we have

$$T^2 v = T(Tv) = T(\lambda v) = \lambda \cdot Tv = \lambda^2 v.$$

Therefore λ^2 is an eigenvalue of T^2. It follows that if either 3 or -3 is an eigenvalue of T, then 9 is an eigenvalue of T^2.

Conversely, suppose that 9 is an eigenvalue of T^2, hence $T^2 - 9I$ is not invertible. Since $T^2 - 9I = (T - 3I)(T + 3I)$ and since a composition of two invertible linear maps is invertible, it follows that either $T - 3I$ or $T + 3I$ is not invertible. Therefore either 3 or -3 is an eigenvalue of T.

(3). (25 points) Find the generalized eigenspaces for the operator $T\colon \mathbf{C}^3 \to \mathbf{C}^3$ represented in the standard basis by the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution. The matrix is in upper-triangular form. In this case we have shown in class that the eigenvalues of T are the entries on the diagonal, namely 1 and 3. We compute separately $G(1, T)$ and $G(3, T)$. Note first that

$$T - I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

and

$$(T - I)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 4 \end{bmatrix}$$

hence $(1, 0, 0)$ and $(0, 1, 0)$ lie in $G(1, T)$. On the other hand, we have

$$T - 3I = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

hence the kernel of $T - 3I$ consists of those (u_1, u_2, u_3) with $-2u_1 + u_2 = 0$, $-2u_2 + 2u_3 = 0$, hence $(1, 2, 2) \in G(3, T)$. Since $(1, 0, 0)$, $(0, 1, 0)$, and $(1, 2, 2)$ give a basis of $\mathbf{C}^3$, we conclude that

$$G(1, T) = \text{span}\big((1, 0, 0), (0, 1, 0)\big) \quad \text{and} \quad G(3, T) = \text{span}\big((1, 2, 2)\big).$$

(4). (25 points) Let V be a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$ an operator that satisfies $T^2 = T$.

i) (5points) Show that the characteristic polynomial of T is equal to $x^m(x-1)^n$ for some nonnegative integers m and n.
ii) (5 points) Show that range$(T) \subseteq E(1,T)$ and null$(T) \subseteq E(0,T)$.
iii) (10 points) Deduce that the inclusions in ii) are, in fact, equalities.
iv) (5 points) Deduce that T is diagonalizable.

Solution. i) The hypothesis says that if $P(x) = x^2 - x$, then $P(T) = 0$. In particular, the minimal polynomial divides P, hence the eigenvalues of T are among 0 and 1. This implies that the characteristic polynomial of T is of the form $x^m(x-1)^n$ for some nonnegative integers m and n.

ii) If $u \in \text{range}(T)$, then we can write $u = Tv$ for some $v \in V$, hence by assumption

$$T(u) = T^2(v) = T(v) = u.$$

Therefore $u \in E(1,T)$.

On the other hand, it follows from definition that $E(0,T) = \text{null}(T)$.

iii) We have already seen that the second inclusion is an equality. Let us show that we also have $E(1,T) \subseteq \text{range}(T)$. If $v \in E(1,T)$, we have $Tv = v$, hence clearly $v \in \text{range}(T)$.

iv) Recall that

$$\dim\big(\text{range}(T)\big) + \dim\big(\text{null}(T)\big) = \dim(V),$$

hence it follows from iii) that

$$\dim\big(E(0,T)\big) + \dim\big(E(1,T)\big) = \dim(V).$$

Since we know that $E(0,T) + E(1,T) = E(0,T) \oplus E(1,T)$, it follows that in fact

$$V = E(0,T) \oplus E(1,T)$$

and we have shown in class that in this case T is diaginalizable.

Alternatively, note that since the minimal polynomial of T divides $x(x-1)$, all its roots appear with multiplicity 1. We have shown in class that in this case T is diaginalizable.