SOFT SHEAVES ON PARACOMPACT SPACES AND APPLICATIONS

1. Sections of a sheaf on closed subsets

Let X be a topological space and $\mathcal{F}$ a sheaf of Abelian groups on X.

Definition 1.1. If Z is a subset of X and $i\colon Z \hookrightarrow X$ is the inclusion map, then

$$\mathcal{F}(Z) := \Gamma(Z, \mathcal{F}|_Z),$$

where $\mathcal{F}|_Z = i^{-1}(Z)$.

Remark 1.2. By definition of $\mathcal{F}|_Z$, it follows that a section s in $\mathcal{F}(Z)$ is given by associating to every $z \in Z$ an element $s(z) \in \mathcal{F}_z$ with the property that for every $x \in Z$, there is an open neighborhood U of x in X and $\widetilde{s} \in \mathcal{F}(U)$ such that $s(z) = \widetilde{s}_z$ for every $z \in U \cap Z$.

Remark 1.3. It is clear that if $Y \subseteq Z$, then we have a canonical restriction map $\mathcal{F}(Z) \to \mathcal{F}(Y)$. Moreover, these restriction maps are functorial in the obvious sense.

Lemma 1.4. *Given a cover $X = Z_1 \cup \ldots \cup Z_r$, with all Z_i closed in X, if $\mathcal{F}$ is a sheaf of Abelian groups on X, then the following sequence induced by restriction maps*

$$0 \to \mathcal{F}(X) \to \prod_i \mathcal{F}(Z_i) \to \prod_{i,j} \mathcal{F}(Z_i \cap Z_j)$$

is exact.

Proof. Consider a family $(s_i)_{1 \leq i \leq r}$, with $s_i \in \mathcal{F}(Z_i)$ such that $s_i|_{Z_i \cap Z_j} = s_j|_{Z_i \cap Z_j}$ for all i and j. We consider each s_i as a map $Z_i \to \bigsqcup_{x \in Z_i} \mathcal{F}_x$ such that $s_i(x) \in \mathcal{F}_x$. The condition on the s_i clearly implies that there is a unique map $s\colon X \to \bigsqcup_{x \in X} \mathcal{F}_x$ such that $s(x) \in \mathcal{F}_x$ for all $x \in X$ and $s(x) = s_i(x)$ for all $x \in Z_i$. We need to show that there is $\widetilde{s} \in \mathcal{F}(X)$ such that $s(x) = \widetilde{s}_x$ for all $x \in X$.

Of course, it is enough to find such $\widetilde{s}$ in the neighborhood of any given point $y \in X$: the resulting sections will glue by the sheaf axiom. Let $y \in X$ be fixed. After possibly replacing X by the complement of those Z_i that do not contain y, we may assume that $y \in Z_1 \cap \ldots \cap Z_r$. After further shrinking X, we may assume that for every i, we have a section $t_i \in \mathcal{F}(X)$ such that $(t_i)_x = s_i(x)$ for all $x \in Z_i$. In particular, we have

$$(t_1)_y = \ldots = (t_r)_y,$$

hence after replacing X one more time by a suitable open neighborhood of y, we may assume that $t_1 = \ldots = t_r$; we may take this section to be $\widetilde{s}$. $\square$

2. Soft sheaves

We now assume that X is a *paracompact* topological space; recall that this means that X is Hausdorff and for every open cover $X=\bigcup_{i\in I}U_i$, there is a locally finite open cover $X=\bigcup_{j\in J}V_j$ that refines it. A useful property is that if $X=\bigcup_{j\in J}V_j$ is a locally finite open cover of a paracompact space X, then there is another open cover $X=\bigcup_{j\in J}W_j$ such that $\overline{W_j}\subseteq V_j$ for all $j\in J$. A special case of this says that if $F\subseteq U$ are subsets of X, with F closed and U open, then there is an open subset V of X such that $F\subseteq V\subseteq\overline{V}\subseteq U$ (this means that X is a *normal space*). We also note that every closed subset of a paracompact space is paracompact: this is easy to see using the definition.

Example 2.1. Every topological manifold (assumed to be Hausdorff and with countable basis of open subsets) is paracompact. Every CW-complex is paracompact.

Definition 2.2. A sheaf of Abelian groups $\mathcal{F}$ on X is *soft* if for every closed subset Z of X, the restriction map

$$\mathcal{F}(X)\to\mathcal{F}(Z)$$

is surjective.

Lemma 2.3. *Let X be a paracompact topological space and $\mathcal{F}$ a sheaf of Abelian groups on X.*

i) *For every closed subset Z of X and every $s\in\mathcal{F}(Z)$, there is an open subset U containing Z and $t\in\mathcal{F}(U)$ such that $t|_Z=s$.*
ii) *If $\mathcal{F}$ is flasque, then it is soft.*

Proof. Let us prove i). By definition of $\mathcal{F}|_Z$, we see that we have a family of open subsets $(U_i)_{i\in I}$, with $Z\subseteq\bigcup_{i\in I}U_i$ and sections $s_i\in\mathcal{F}(U_i)$ such that $s_i|_{U_i\cap Z}=s|_{U_i\cap Z}$ for all $i\in I$. We consider the cover of X by the U_i and by $X\smallsetminus Z$; after passing to a suitable refinement, we may assume that this a locally finite cover. We can then find open subsets V_i, with $\overline{V_i}\subseteq U_i$, and such that $Z\subseteq\bigcup_{i\in I}V_i$.

Given any $x\in Z$, we choose an open neighborhood $U(x)$ of x that intersects only finitely many of the U_j and such that $U(x)$ is contained in some V_i. We put $s^{(x)}=s_i|_{U(x)}$. In particular, $s^{(x)}$ and s take the same value at x. If $x\not\in\overline{V_j}$ for some j, we may replace $U(x)$ by $U(x)\smallsetminus\overline{V_j}$. Since $U(x)$ intersects only finitely many U_j, it follows that after repeating this operation finitely many times, we may assume that whenever $U(x)\cap\overline{V_j}\neq\emptyset$, we have $x\in\overline{V_j}\subseteq U_j$. After further shrinking $U(x)$, we may thus assume, in addition, that for such j we have $U(x)\subseteq U_j$. Since $s^{(x)}$ and s_j take the same value at x, after further shrinking $U(x)$, we may assume that for all such j, we have $s^{(x)}=s_j|_{U(x)}$.

We put $U=\bigcup_{x\in Z}U(x)$. It is clear that U is an open neighborhood of Z. We claim that

$$s^{(x)}|_{U(x)\cap U(y)}=s^{(y)}|_{U(x)\cap U(y)}\quad\text{for all}\quad x,y\in Z. \tag{1}$$

If $z \in U(x) \cap U(y)$, then $z \in V_\ell$, for some ℓ. Since $U(x) \cap \overline{V_\ell} \neq \emptyset$ and $U(y) \cap \overline{V_\ell} \neq \emptyset$, then by construction we have $U(x), U(y) \subseteq U_\ell$ and

$$s_\ell|_{U(x)} = s^{(x)} \quad \text{and} \quad s_\ell|_{U(y)} = s^{(y)},$$

which gives (1). We can thus find $t \in \mathcal{F}(U)$ such that $t|_{U(x)} = s^{(x)}$ for all $x \in Z$. In particular, we have $t_x = s_x$ for every $x \in Z$, and thus $t|_Z = s$.

In order to prove the assertion in ii), suppose that $\mathcal{F}$ is flasque. Given a closed subset of Z of X and $s \in \mathcal{F}(Z)$, we use i) to find an open neighborhood U of Z and $t \in \mathcal{F}(U)$ such that $t|_Z = s$. Since $\mathcal{F}$ is flasque, it follows that there is $t' \in \mathcal{F}(X)$ such that $t'|_U = t$, hence $t'|_Z = s$. Therefore $\mathcal{F}$ is soft. □

Lemma 2.4. *Let X be a paracompact topological space. Given a short exact sequence of sheaves of Abelian groups*

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \longrightarrow 0,$$

with $\mathcal{F}'$ soft, the corresponding sequence of global sections

$$0 \longrightarrow \mathcal{F}'(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}''(X) \longrightarrow 0$$

is exact.

Proof. We only need to prove that for every section $s'' \in \mathcal{F}''(X)$, there is $s \in \mathcal{F}(X)$ such that $\psi(s) = s''$. By definition, we can find an open cover $X = \bigcup_{i\in I} U_i$ and sections $s_i \in \mathcal{F}(U_i)$ such that $\psi(s_i) = s|_{U_i}$ for all i. After passing to a refinement, we may assume that the cover is locally finite. We may now find another cover $X = \bigcup_{i\in I} V_i$ such that $\overline{V_i} \subseteq U_i$ for all $i \in I$.

For every $J \subseteq I$, put $Z_J = \bigcup_{i\in J} \overline{V_i}$. We consider the set $\mathcal{P}$ of pairs (J, t), where $J \subseteq I$ and $t \in \mathcal{F}(Z_J)$ is such that $\psi(t) = s|_{Z_J}$. We order it by $(J_1, t_1) \leq (J_2, t_2)$ if $J_1 \subseteq J_2$ and $t_2|_{Z_{J_1}} = t_1$. It is straightforward to see that we may apply Zorn's lemma to choose a maximal element (J, s) of $\mathcal{P}$. If $J = I$, then $Z_J = X$, and $\psi(s) = s''$.

Suppose now that $J \neq I$ and let $i \in I \smallsetminus J$. Since $\psi(s_i|_{\overline{V_i}\cap Z_J}) = \psi(s|_{\overline{V_i}\cap Z_J})$, it follows that

$$s_i|_{\overline{V_i}\cap Z_J} - s|_{\overline{V_i}\cap Z_J} = \varphi(s'),$$

for some $s' \in \mathcal{F}'(\overline{V_i} \cap Z_J)$. Since $\mathcal{F}'$ is soft, we can find $v \in \mathcal{F}'(X)$ such that $v|_{\overline{V_i}\cap Z_J} = s'$. After replacing s_i by $s_i - \varphi(v|_{U_i})$, we may thus assume that $s_i|_{\overline{V_i}\cap Z_J} = s|_{\overline{V_i}\cap Z_J} = \varphi(s')$, hence by Lemma 1.4 we can find a section $t' \in \mathcal{F}(Z_{J\cup\{i\}})$ such that $t'|_{Z_J} = t$ and $t'|_{\overline{V_i}} = s_i|_{\overline{V_i}}$. In this case $\psi(t') = s''|_{Z_{J\cup\{i\}}}$, contradicting the maximality of J. □

Lemma 2.5. *If X is a paracompact topological space and we have a short exact sequence of sheaves of Abelian groups on X*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

with $\mathcal{F}'$ soft, then $\mathcal{F}$ is soft if and only if $\mathcal{F}''$ is soft.

Proof. If Z is a closed subset of X, then we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}'(X) & \longrightarrow & \mathcal{F}(X) & \longrightarrow & \mathcal{F}''(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}'(Z) & \longrightarrow & \mathcal{F}(Z) & \longrightarrow & \mathcal{F}''(Z) & \longrightarrow & 0. \end{array}$$

The rows are exact by Lemma 2.4 (note that if $\mathcal{F}'$ is soft, then clearly $\mathcal{F}'|_Z$ is soft, and Z is paracompact, being closed in X). Moreover, since $\mathcal{F}'$ is soft, the first vertical map is surjective, hence the second one is surjective if and only if the third one is. $\square$

Proposition 2.6. *If X is a paracompact topological space and $\mathcal{E}$ is a soft sheaf of Abelian groups on X, then*

$$H^i(X, \mathcal{E}) = 0 \quad \textit{for all} \quad i \geq 1.$$

In particular, if $\mathcal{F}$ has a resolution $\mathcal{F} \to \mathcal{E}^\bullet$, with all $\mathcal{E}^i$ soft sheaves, then we have a canonical isomorphism

$$H^i(X, \mathcal{F}) \simeq \mathcal{H}^i\big(\Gamma(X, \mathcal{E}^\bullet)\big).$$

Proof. We argue by induction on $i \geq 1$. Consider a short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{E} \to \mathcal{A} \to \mathcal{B} \to 0,$$

with $\mathcal{A}$ flasque. By Lemma 2.3, we see that $\mathcal{A}$ is soft, hence $\mathcal{B}$ is soft by Lemma 2.5. The long exact sequence in cohomology for the above short exact sequence gives

$$0 \to \Gamma(X, \mathcal{E}) \to \Gamma(X, \mathcal{A}) \overset{\alpha}{\to} \Gamma(X, \mathcal{B}) \to H^1(X, \mathcal{E}) \to H^1(X, \mathcal{A}).$$

Note that the map α is surjective by Lemma 2.4, and since $\mathcal{A}$ is flasque, we have $H^i(X, \mathcal{A}) = 0$ for $i \geq 1$. First, we conclude that $H^1(X, \mathcal{E}) = 0$, completing the proof of the case $i = 1$ in the induction.

Moreover, the long exact sequence in cohomology gives isomorphisms

$$H^i(X, \mathcal{E}) \simeq H^{i-1}(X, \mathcal{B}) \quad \text{for all} \quad i \geq 2.$$

Since $\mathcal{B}$ is soft, we have $H^{i-1}(X, \mathcal{B}) = 0$ by induction, and thus $H^i(X, \mathcal{E}) = 0$.

The last assertion in the proposition is now a direct consequence of the fact that cohomology can be computed by acylic resolutions. $\square$

Remark 2.7. Suppose that $f\colon (X', \mathcal{O}_{X'}) \to (X, \mathcal{O}_X)$ is a morphism of ringed spaces, with both X and X' paracompact, $\mathcal{F}$ is an $\mathcal{O}_X$-module, $\mathcal{F}'$ is an $\mathcal{O}_{X'}$-module, and we have a morphism $f^*(\mathcal{F}) \to \mathcal{F}'$ inducing maps

$$H^q(X, \mathcal{F}) \to H^q(X', \mathcal{F}'). \tag{2}$$

If $\mathcal{F} \to \mathcal{E}^\bullet$ and $\mathcal{F}' \to \mathcal{E}'^\bullet$ are resolutions by soft sheaves and we have morphisms $f^*(\mathcal{E}^i) \to \mathcal{E}'^i$ that commute with the differentials in the two complexes, we obtain morphisms

$$\mathcal{H}^i\big(\Gamma(X, \mathcal{E}^\bullet)\big) \to \mathcal{H}^i\big(\Gamma(X', \mathcal{E}'^\bullet)\big). \tag{3}$$

Then the morphisms (2) and (3) coincide via the isomorphisms provided by Proposition 2.6. We leave the proof as an exercise for the reader.

3. Soft sheaves on smooth manifolds

In this section we consider a smooth manifold M. Recall that by assumption M is assumed to be Hausdorff and with a countable basis of open subsets, hence it is paracompact. Let $n = \dim(M)$. We denote by $\mathcal{C}_M^\infty$ the sheaf of smooth functions on X with values in $\mathbf{R}$. For every $p \geq 0$, the sheaf $\mathcal{A}_M^p$ of smooth p-forms on M is a $\mathcal{C}_M^\infty$-module. Note that $\mathcal{A}_M^0 = \mathcal{C}_M^\infty$ and $\mathcal{A}_M^p = 0$ if $p > n$.

Proposition 3.1. *Every $\mathcal{C}_M^\infty$-module is a soft sheaf.*

Proof. Let $\mathcal{F}$ be a $\mathcal{C}_M^\infty$-module and let $s \in \mathcal{F}(Z)$, where Z is a closed subset of X. By assertion i) in Lemma 2.3, there is an open subset U of X, containing Z, and $t \in \mathcal{F}(U)$, such that $t|_Z = s$. Let us choose open subsets U_1 and U_2 such that

$$Z \subseteq U_1 \subseteq \overline{U_1} \subseteq U_2 \subseteq \overline{U_2} \subseteq U.$$

Since X is a smooth manifold, by the smooth version of Urysohn's lemma, we can find a global section $f \in \mathcal{C}_X^\infty(X)$ such that $f = 1$ on $\overline{U_1}$ and $f = 0$ on $X \smallsetminus U_2$. Since $\mathcal{F}$ is a sheaf and $(ft)|_{U \smallsetminus \overline{U_2}} = 0$, we can find $v \in \mathcal{F}(X)$ such that $v|_U = ft$ and $v|_{X \smallsetminus \overline{U_2}} = 0$. It is clear that $v|_{U_1} = t|_{U_1}$, hence $v|_Z = s$. □

4. Application to Dolbeault cohomology

Suppose now that X is a complex manifold of dimension n. We have seen that for every p, we have an exact complex

$$0 \longrightarrow \Omega_X^p \to \mathcal{A}_X^{p,0} \xrightarrow{\overline{\partial}} \mathcal{A}_X^{p,1} \xrightarrow{\overline{\partial}} \ldots \xrightarrow{\overline{\partial}} \mathcal{A}_X^{p,n} \to 0. \tag{4}$$

Note that the sheaves $\mathcal{A}_X^{p,q}$ are in particular $\mathcal{C}_X^\infty$-modules, hence they are soft by Proposition 3.1.

The *Dolbeault cohomology group of* X, denoted $H^{p,q}(X)$, is defined as the q^{th} cohomology of the complex

$$0 \to \Gamma(X, \mathcal{A}_X^{p,0}) \xrightarrow{\overline{\partial}} \Gamma(X, \mathcal{A}_X^{p,1}) \xrightarrow{\overline{\partial}} \ldots \xrightarrow{\overline{\partial}} \Gamma(X, \mathcal{A}_X^{p,n}) \longrightarrow 0.$$

The following result now follows now from Proposition 2.6.

Theorem 4.1. *For every complex manifold X and every p and q, we have an isomorphism*

$$H^{p,q}(X) \simeq H^q(X, \Omega_X^p).$$

More generally, consider a holomorphic vector bundle E, with sheaf of holomorphic sections $\mathcal{E}$. Note that the sheaf of smooth sections of E is $\mathcal{E}_{\text{sm}} = \mathcal{C}_X^\infty \otimes_{\mathcal{O}_X} \mathcal{E}$. If we put

$$\mathcal{A}_{X,\mathcal{E}}^{p,q} := \mathcal{A}_X^{p,q} \otimes_{\mathcal{C}_X^\infty} \mathcal{E}_{\text{sm}} = \mathcal{A}_X^{p,q} \otimes_{\mathcal{O}_X} \mathcal{E},$$

then we have an exact complex

$$0 \longrightarrow \Omega_X^p \otimes_{\mathcal{O}_X} \mathcal{E} \longrightarrow \mathcal{A}_{X,\mathcal{E}}^{p,0} \xrightarrow{\overline{\partial}_{\mathcal{E}}} \ldots \xrightarrow{\overline{\partial}_{\mathcal{E}}} \mathcal{A}_{X,\mathcal{E}}^{p,n} \longrightarrow 0 \tag{5}$$

obtained by tensoring (4) with $\mathcal{E}$. The *Dolbeault cohomology group of X with coefficients in $\mathcal{E}$*, denoted $H^{p,q}(X,\mathcal{E})$ is the q^{th} cohomology of the induced complex:

$$0 \to \Gamma(X, \mathcal{A}^{p,0}_{X,\mathcal{E}}) \xrightarrow{\overline{\partial}_{\mathcal{E}}} \Gamma(X, \mathcal{A}^{p,1}_{X,\mathcal{E}}) \xrightarrow{\overline{\partial}_{\mathcal{E}}} \ldots \xrightarrow{\overline{\partial}_{\mathcal{E}}} \Gamma(X, \mathcal{A}^{p,n}_{X,\mathcal{E}}) \longrightarrow 0.$$

The sheaves $\mathcal{A}^{p,q}_{X,\mathcal{E}}$ are again $\mathcal{C}^{\infty}_X$-modules, hence soft by Proposition 3.1; we deduce the following result via Proposition 2.6.

Theorem 4.2. *For every complex manifold X, every locally free $\mathcal{O}_X$-module $\mathcal{E}$ of finite rank, and every p and q, we have a canonical isomorphism*

$$H^{p,q}(X,\mathcal{E}) \simeq H^q(X, \Omega^p_X \otimes_{\mathcal{O}_X} \mathcal{E}).$$

Remark 4.3. It is clear that both $H^{p,q}(X,-)$ and $H^q(X,\Omega^p_X \otimes_{\mathcal{O}_X} -)$ give functors on locally free $\mathcal{O}_X$-modules of finite rank and the isomorphism provided by Theorem 4.2 is functorial.

Remark 4.4. Suppose that $f\colon X \to Y$ is a holomorphic map between complex manifolds and $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank. The canonical morphism $f^*\Omega^p_Y \to \Omega^p_X$ induces a morphism

$$f^*(\Omega^p_Y \otimes_{\mathcal{O}_Y} \mathcal{E}) \to \Omega^p_X \otimes_{\mathcal{O}_X} f^*(\mathcal{E})$$

and thus canonical $\mathbf{C}$-linear maps

$$H^q(Y, \Omega^p_Y \otimes_{\mathcal{O}_Y} \mathcal{E}) \to H^q\big(X, \Omega^p_X \otimes_{\mathcal{O}_X} f^*(\mathcal{E})\big).$$

On the other hand, by pulling back (p,q)-forms, we obtain $\mathbf{C}$-linear maps

$$\Gamma(Y, \mathcal{A}^{p,q}_Y \otimes_{\mathcal{O}_Y} \mathcal{E}) \to \Gamma\big(X, \mathcal{A}^{p,q}_X \otimes_{\mathcal{O}_X} f^*(\mathcal{E})\big)$$

which commute with the differentials in the two complexes and thus induce $\mathbf{C}$-linear maps

$$H^{p,q}(Y,\mathcal{E}) \to H^{p,q}\big(X, f^*(\mathcal{E})\big).$$

It follows from Remark 2.7 that the diagram

$$\begin{array}{ccc} H^q(Y, \Omega^p_Y \otimes_{\mathcal{O}_Y} \mathcal{E}) & \longrightarrow & H^q\big(X, \Omega^p_X \otimes_{\mathcal{O}_X} f^*(\mathcal{E})\big) \\ \downarrow & & \downarrow \\ H^{p,q}(Y,\mathcal{E}) & \longrightarrow & H^{p,q}\big(X, f^*(\mathcal{E})\big), \end{array}$$

in which the vertical maps are the isomorphisms provided by Theorem 4.2, is commutative.

5. The De Rham theorem

A more basic result concerns the description of $H^q(M,\mathbf{R})$ as De Rham cohomology for a smooth manifold M. For such M, let $n = \dim(M)$ and consider the complex defined by exterior differentiation:

$$0 \to \mathcal{A}^0_M \xrightarrow{d} \mathcal{A}^1_M \xrightarrow{d} \ldots \xrightarrow{d} \mathcal{A}^n_M \longrightarrow 0. \tag{6}$$

The *De Rham* cohomology of M, denoted $H^q_{\mathrm{DR}}(M,\mathbf{R})$ is the q^{th} cohomology of the induced complex obtained by taking global sections:

$$0 \to \Gamma(M,\mathcal{A}^0_M) \xrightarrow{d} \Gamma(M,\mathcal{A}^1_M) \xrightarrow{d} \ldots \xrightarrow{d} \Gamma(M,\mathcal{A}^n_M) \longrightarrow 0.$$

Theorem 5.1. *For every smooth manifold M, we have an isomorphism*

$$H^*_{\mathrm{DR}}(M,\mathbf{R}) \simeq H^*(M,\mathbf{R}),$$

where on the right-hand side we have the sheaf cohomology with coefficients in the constant sheaf $\mathbf{R}$.

Proof. It is straightforward to see that the kernel of

$$\mathcal{A}^0_M \xrightarrow{d} \mathcal{A}^1_M$$

is the constant sheaf $\mathbf{R}$ on M. Moreover, every sheaf $\mathcal{A}^p_M$ is a $\mathcal{C}^\infty_M$-module, hence it is soft by Proposition 3.1. The isomorphism in Theorem 5.1 thus follows from Proposition 2.6 if we show that the complex in (6) has no cohomology in positive degrees. This follows in the same way as the $\overline{\partial}$-lemma (but it is easier); we leave the proof as an exercise for the reader. □

Remark 5.2. Recall that the exterior product of differential forms satisfies

$$d(\omega\wedge\eta) = d(\omega)\wedge\eta + (-1)^p\omega\wedge d(\eta) \quad \text{for} \quad \omega\in\Gamma(M,\mathcal{A}^p_M),\ \eta\in\Gamma(M,\mathcal{A}^q_M).$$

It is thus immediate that the exterior product induces a multiplication on

$$H^*_{\mathrm{DR}}(M,\mathbf{R}) = \bigoplus_{p=0}^{n} H^p_{\mathrm{DR}}(M,\mathbf{R})$$

that makes this a graded-commutative $\mathbf{R}$-algebra. One can show that the isomorphism in Theorem 5.1 is an isomorphism of graded-commutative $\mathbf{R}$-algebras.

Remark 5.3. We have a similar description of $H^*(M,\mathbf{C})$ if we work with differential forms with complex coefficients, by tensoring the complex (6) with $\mathbf{C}$.

Remark 5.4. We have an analogue of the assertion in Remark 4.4 in the case of a smooth map $f\colon M'\to M$.