Codimension one foliations on projective manifolds

Lecture 1 — Closed meromorphic 1-forms and index theorems

Jorge Vitório Pereira
IMPA

Ann Arbor, May 2026

Simons Collaboration on Moduli, Ann Arbor

The plan

Four lectures.

L1. Closed meromorphic 1-forms; index theorems for invariant hypersurfaces.

L2. Existence of separatrices on surfaces (Camacho–Sad).

L3. Structure of codimension-one foliations on projective manifolds: transversely projective foliations and the Cerveau–Lins Neto conjecture.

L4. Reduction to positive characteristic and the conjecture and other topics.

Main references:

- J. V. Pereira, *Closed meromorphic 1-forms*, in Handbook of Geometry and Topology of Singularities V: Foliations, Springer, Cham, 2024, pp. 447–499.
- F. Loray, J. V. Pereira, F. Touzet, *Deformation of rational curves along foliations*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 21 (2020), 1315–1331.
- W. Mendson, J. V. Pereira, *Codimension one foliations in positive characteristic*, J. Inst. Math. Jussieu 23 (2024), no. 4, 1705–1750.
- J. V. Pereira, *Codimension one foliations on projective manifolds*, Notices Amer. Math. Soc. 71 (2024), no. 8, 1025–1031.

Today

- Codimension-one foliations: definitions.
- Closed meromorphic 1-forms and their residues.
- Weil's residue theorem: $c(\operatorname{Res}\omega) = 0$.
- Meromorphic flat connections on line bundles; residue theorem.
- Bott's connection along an invariant hypersurface.
- Camacho–Sad index theorem for smooth invariant hypersurfaces.
- Jouanolou's theorem: very general foliation on $\mathbb{P}^2$ has no algebraic leaf.

Codim-one foliations: setup

What is a foliation?

X a connected complex manifold. A codim-q singular holomorphic foliation $\mathcal{F}$ on X is a pair

$$(\mathcal{T}_{\mathcal{F}}, \mathcal{N}^*_{\mathcal{F}}) \subset (T_X, \Omega^1_X)$$

of coherent subsheaves, each the annihilator of the other, with $\mathcal{T}_{\mathcal{F}}$ involutive and of generic rank $\dim X - q$.

Equivalently, the data of a saturated rank-q subsheaf $\mathcal{N}^*_{\mathcal{F}} \subset \Omega^1_X$ satisfying $\bigwedge^q \mathcal{N}^*_{\mathcal{F}} \wedge d\mathcal{N}^*_{\mathcal{F}} = 0$ in Ω^{q+2}_X.

Codimension one

$\mathcal{N}^*_{\mathcal{F}}$ is reflexive of rank one $\Rightarrow$ locally free $\Rightarrow$ a line bundle, with dual $\mathcal{N}_{\mathcal{F}}$.

Globally, $\mathcal{F}$ is determined by a twisted 1-form

$$\omega \in H^0\big(X, \Omega^1_X \otimes \mathcal{N}_{\mathcal{F}}\big), \qquad \omega \wedge d\omega = 0,$$

modulo multiplication by global nowhere-vanishing holomorphic functions on X.

The zero locus of ω has codimension ≥ 2.

Singular set, invariant subvarieties

Singular set. $\mathrm{sing}(\mathcal{F})$ = locus where $T_X/\mathcal{T}_{\mathcal{F}}$ is not locally free. It has codimension ≥ 2. In codimension one, equal to the zero set of ω.

Invariant subvariety. $Y \subset X$ irreducible is $\mathcal{F}$-invariant if $Y \not\subset \mathrm{sing}(\mathcal{F})$ and any local generator ω of $\mathcal{N}^*_{\mathcal{F}}$ pulls back to zero on $Y_{\mathrm{smooth}} - \mathrm{sing}(\mathcal{F})$.

Separatrix. Germ at $p \in \mathrm{sing}(\mathcal{F})$ of a subvariety S with $S - \mathrm{sing}(\mathcal{F})$ equal to a germ of a leaf.

Closed meromorphic 1-forms

Foliations from closed meromorphic 1-forms

Let ω be a closed meromorphic 1-form on X, $D = (\omega)_\infty - (\omega)_0$.

The foliation $\mathcal{F}_\omega$ has $\mathcal{T}_{\mathcal{F}_\omega} = \ker\big(T_X \xrightarrow{\omega} \mathcal{O}_X(D)\big)$ and $\mathcal{N}^*_{\mathcal{F}_\omega} = \mathcal{O}_X(-D)$.

Lemma
Every irreducible component of the divisor $(\omega)_0 \cup (\omega)_\infty$ is invariant by $\mathcal{F}_\omega$.

Proof. If $\eta = f\omega$ generates $\mathcal{N}^*_{\mathcal{F}}$, then $d\eta = \frac{df}{f} \wedge \eta$. Any component of the support of $\mathrm{div}(f) = -D$ is invariant. $\square$

Residues of a closed meromorphic 1-form

$H \subset X$ irreducible hypersurface, $p \in H$ smooth, $i \colon \overline{\mathbb{D}} \hookrightarrow X$ a small disc meeting H transversally at p.

$$\mathrm{Res}_H(\omega) \;:=\; \frac{1}{2\pi\sqrt{-1}} \int_{\partial\mathbb{D}} i^*\omega \;\in\; \mathbb{C}.$$

Independent of p and i, by closedness of ω.

Residue divisor.

$$\mathrm{Res}(\omega) = \sum_H \mathrm{Res}_H(\omega) \cdot H \;\in\; \mathrm{Div}(X) \otimes \mathbb{C}.$$

Theorem (Weil 1947)
X a (not necessarily compact) complex manifold, ω a closed meromorphic 1-form on X. Then

$$c(\mathrm{Res}(\omega)) = 0 \quad \text{in } H^2(X, \mathbb{C}).$$

For compact curves, intersecting with $[X]$ recovers the familiar identity $\sum \mathrm{Res}_p(\omega) = 0$ for closed meromorphic 1-forms on compact Riemann surfaces.

Proof of Weil's theorem

Pick $\sigma \in H_2(X, \mathbb{Z})$. Represent $\sigma = \sum \sigma_j$ by 2-simplices meeting $|(\omega)_\infty|$ transversally at single smooth points (or not at all).

If σ_j meets $H_i \subset |\operatorname{Res}(\omega)|$, then $\partial\sigma_j \sim (H_i \cdot \sigma_j)\,\gamma_i$ on $X - |(\omega)_\infty|$ with γ_i the boundary of a small transversal disc.

$$\operatorname{Res}(\omega) \cdot \sigma \;=\; \frac{1}{2\pi i} \sum_{\sigma_j \cap |\operatorname{Res}(\omega)| \neq \emptyset} \int_{\partial\Delta^2} \sigma_j^* \omega \;=\; -\frac{1}{2\pi i} \sum_{\sigma_j \cap |\operatorname{Res}(\omega)| = \emptyset} \int_{\partial\Delta^2} \sigma_j^* \omega \;=\; 0.$$

The last step uses $\int_{\partial\Delta^2} \sigma_j^* \omega = \int_{\Delta^2} d\sigma_j^* \omega = 0$ since ω is closed off its polar set. $\square$

Flat connections and the residue theorem

Meromorphic connections on line bundles

L a line bundle on X, D a divisor. A meromorphic connection on L with poles on D is a $\mathbb{C}$-linear map of sheaves

$$\nabla\colon L \longrightarrow \Omega^1_X(*D) \otimes L$$

satisfying Leibniz, $\nabla(f\sigma) = df \otimes \sigma + f\nabla(\sigma)$.

In a trivialisation $\{\sigma_i\}$ with cocycle g_{ij}: $\nabla(\sigma_i) = \eta_i \otimes \sigma_i$ and $\eta_j - \eta_i = \mathrm{dlog}\, g_{ij}$.

Curvature. $\Theta = \frac{1}{2\pi i}\, d\eta_i$, globally defined meromorphic 2-form. ∇ is flat if $\Theta \equiv 0$.

Residue theorem for flat connections

∇ flat $\Rightarrow$ the local 1-forms η_i are closed $\Rightarrow$ residues $\mathrm{Res}_H(\nabla)$ are well-defined complex numbers, and a divisor

$$\mathrm{Res}(\nabla) = \sum_H \mathrm{Res}_H(\nabla)\, H.$$

Theorem
For $L \in \mathrm{Pic}(X)$ and any flat meromorphic connection ∇ on L,

$$c(L) \ = \ -c(\mathrm{Res}(\nabla)) \quad \text{in } H^2(X, \mathbb{C}).$$

Idea. On the total space $E(L) \xrightarrow{\pi} X$, ∇ defines a closed meromorphic 1-form ω with $\mathrm{Res}(\omega) = X + \pi^* \mathrm{Res}(\nabla)$. Apply Weil to ω and restrict to/pullback by the zero section.

Bott's connection and Camacho–Sad

Bott's partial connection

$\mathcal{F}$ codim-one on X, $Y \subset X$ smooth $\mathcal{F}$-invariant hypersurface, ω_i local generators of $\mathcal{N}^*_{\mathcal{F}}$, $\omega_i = g_{ij}\omega_j$.

Find η_i with $d\omega_i = \eta_i \wedge \omega_i$ and $(\eta_i)_\infty \not\supset Y$.

Then $\iota^*\eta_i - \iota^*\eta_j = \mathrm{dlog}\, \iota^* g_{ij}$ on Y and $d\iota^*\eta_i = 0$.

So $\{\iota^*\eta_i\}$ defines a flat meromorphic connection ∇^B on $\mathcal{N}^*_{\mathcal{F}}|_Y$.

A worked example

$\mathcal{F}$ on $(\mathbb{C}^3, 0)$ defined by

$$\omega = xyz\left(\alpha\frac{dx}{x} + \beta\frac{dy}{y} + \gamma\frac{dz}{z}\right), \qquad \alpha, \beta, \gamma \in \mathbb{C}^*.$$

$Y = \{x = 0\}$ is invariant. Then $d\omega = \mathrm{dlog}(xyz) \wedge \omega$, so picking $\eta_\mu = \mathrm{dlog}(xyz) + \mu(xyz)^{-1}\omega$ with $\mu = -\alpha^{-1}$,

$$\nabla^B|_Y \;=\; d \;+\; \left(1 - \tfrac{\beta}{\alpha}\right)\frac{dy}{y} \;+\; \left(1 - \tfrac{\gamma}{\alpha}\right)\frac{dz}{z}.$$

Polar set: $\{y = 0\} \cup \{z = 0\} \subset Y$.

Variation

If S is an irreducible component of $\mathrm{sing}(\mathcal{F}) \cap Y$,

$$\mathrm{Var}(\mathcal{F}, Y, S) \;:=\; \mathrm{Res}_S(\nabla^B).$$

Suwa's variation. Residue of Bott's connection along S.

In the example: $\mathrm{Var}(\mathcal{F}, Y, \{y = 0\}) = 1 - \beta/\alpha$, $\mathrm{Var}(\mathcal{F}, Y, \{z = 0\}) = 1 - \gamma/\alpha$.

Residue theorem on Y gives

$$c(\mathcal{N}_{\mathcal{F}}|_Y) \;=\; \sum_S \mathrm{Var}(\mathcal{F}, Y, S)\, c(S) \quad \text{in } H^2(Y, \mathbb{C}).$$

Camacho–Sad index

Write $\omega = h\,df + f\,\eta$ near $Y = \{f = 0\}$, with $h \in \mathcal{O}_X$ and $\eta \in \Omega^1_X$. Then $\mathrm{sing}(\mathcal{F}) \cap Y = \{f = h = 0\}$.

Integrability $\omega \wedge d\omega = 0 \Rightarrow \iota^*(\eta/h)$ is a closed meromorphic 1-form on Y.

$$\mathrm{CS}(\mathcal{F}, Y, S) \;:=\; \mathrm{Res}_S\big(\iota^*(\eta/h)\big) \;\in\; \mathbb{C}.$$

Camacho–Sad index along S.

Var, CS and the zero divisor

$\omega|_Y \in H^0(Y, \Omega^1_X|_Y \otimes \mathcal{N}_{\mathcal{F}}|_Y)$ lies inside the image of $\mathcal{N}_Y^* \otimes \mathcal{N}_{\mathcal{F}}|_Y$. Its zero divisor on Y is $Z(\mathcal{F}, Y) = \sum_S Z(\mathcal{F}, Y, S)\, S$.

Proposition (Brunella)

$$\mathrm{Var}(\mathcal{F}, Y, S) \;=\; \mathrm{CS}(\mathcal{F}, Y, S) + Z(\mathcal{F}, Y, S).$$

And on Y: $\mathcal{N}_Y \;=\; \mathcal{N}_{\mathcal{F}}|_Y \otimes \mathcal{O}_X(-Z(\mathcal{F}, Y))$.

Camacho–Sad index theorem

Theorem (Camacho–Sad)
Let $\mathcal{F}$ be a codim-one foliation on X and Y a smooth $\mathcal{F}$-invariant hypersurface. Then

$$c(\mathcal{N}_Y) \;=\; \sum_S \mathrm{CS}(\mathcal{F}, Y, S)\, c(S) \quad \text{in } H^2(Y, \mathbb{C}),$$

where the sum runs over the irreducible components of $\mathrm{sing}(\mathcal{F}) \cap Y$.

Proof. $c(\mathcal{N}_Y) = c(\mathrm{Var}(\mathcal{F}, Y)) - c(Z(\mathcal{F}, Y)) = c(\mathrm{CS}(\mathcal{F}, Y))$. □

The identity above realises the Camacho–Sad index as the residue divisor of a flat meromorphic connection ∇^{CS} on $\mathcal{N}_Y$, dual to the tensor product $(\mathcal{N}^*_{\mathcal{F}}|_Y, \nabla^B) \otimes (\mathcal{O}_Y(Z(\mathcal{F}, Y)), \nabla^Z)$.

The case of surfaces

$\dim X = 2$, $Y = C \subset X$ smooth $\mathcal{F}$-invariant curve. Then $\mathrm{sing}(\mathcal{F}) \cap C$ is a finite set $\{p_1, \ldots, p_r\}$ and

$$C \cdot C \;=\; \sum_{i=1}^{r} \mathrm{CS}(\mathcal{F}, C, p_i).$$

This is the original 1982 Camacho–Sad index theorem.

In Lecture 2 we will use this identity, together with reduction of singularities on surfaces, to establish the existence of separatrices.

Some consequences

(i) If C is a smooth invariant curve on a surface X with $C^2 > 0$, then $\mathcal{F}$ has at least one singularity on C with non-zero Camacho–Sad index.

(ii) For a line $\ell \subset \mathbb{P}^2$ invariant by a foliation of degree d, the sum of the Camacho–Sad indices of $\mathcal{F}$ along ℓ equals 1.

(iii) When $\mathcal{N}_{\mathcal{F}}$ is positive enough and Y is a smooth invariant hypersurface, the identity bounds the topological complexity of $\mathrm{sing}(\mathcal{F}) \cap Y$.

Two refinements used later in the course

Simple singularities. A closed meromorphic 1-form with simple singularities admits, near any point of its polar set, a local expression of the form

$$\omega = \sum_{i=1}^{n} \lambda_i \frac{dx_i}{x_i} + d\left(\frac{1}{x_1^{m_1} \cdots x_n^{m_n}}\right).$$

Any closed meromorphic 1-form becomes, after a proper bimeromorphic modification, a closed meromorphic 1-form with simple singularities.

Hodge theory. On a compact Kähler manifold X, every logarithmic 1-form with simple normal crossing poles is d-closed (Deligne), and a divisor $D \in \mathrm{Div}(X) \otimes \mathbb{C}$ is the residue divisor of a closed logarithmic 1-form if and only if $c(D) = 0$ (Weil).

Jouanolou's theorem

Jouanolou's theorem

Theorem (Jouanolou, 1979)
A generic foliation of degree $d \geq 2$ on $\mathbb{P}^2$ admits no invariant algebraic curve.

Setup. A foliation $\mathcal{F}$ of degree d on $\mathbb{P}^2$ is given by

$$\Omega = A\,dx + B\,dy + C\,dz, \qquad A, B, C \in \mathbb{C}[x, y, z]_{d+1},$$

with $xA + yB + zC = 0$. Write $\Sigma_d = H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(d+2))$ and $\mathrm{Fol}(d) = \mathbb{P}(\Sigma_d)$; $\dim_{\mathbb{C}} \Sigma_d = d^2 + 4d + 3$.

A reduced curve $\{F = 0\} \subset \mathbb{P}^2$ is $\mathcal{F}$-invariant iff there exists a polynomial 2-form Θ of degree d with

$$\Omega \wedge dF \;=\; F \cdot \Theta.$$

Reduction to Baire

For $k \geq 1$, set

$$\mathcal{C}_k(d) = \{\, \mathcal{F} \in \mathrm{Fol}(d) : \mathcal{F} \text{ has an invariant curve of degree } k \,\}.$$

Eliminating Θ from $\Omega \wedge dF = F\,\Theta$ exhibits $\mathcal{C}_k(d)$ as the projection of a closed algebraic subset of $\mathbb{P}^2 \times \mathbb{P}(\Sigma_d \times \Lambda_d) \times \mathbb{P}(S_k)$; hence $\mathcal{C}_k(d)$ is Zariski-closed in $\mathrm{Fol}(d)$.

By Baire it suffices to show $\mathcal{C}_k(d) \neq \mathrm{Fol}(d)$ for every $k \geq 1$; the complement of $\bigcup_k \mathcal{C}_k(d)$ is then a dense G_δ.

Universal singular set.

$$S(d) = \{(x, \mathcal{F}) \in \mathbb{P}^2 \times \mathrm{Fol}(d) : x \in \mathrm{sing}(\mathcal{F})\}$$

is irreducible of codimension two, by transitivity of PGL_3 on $\mathbb{P}(T\mathbb{P}^2)$.

Localization at a singularity

Lemma (Localization)
Every $\mathcal{F}$-invariant algebraic curve $C \subset \mathbb{P}^2$ meets $\mathrm{sing}(\mathcal{F})$.

Immediate from Camacho–Sad: were C smooth and disjoint from $\mathrm{sing}(\mathcal{F})$, the index sum would be empty and $C^2 = 0$, contradicting $C^2 > 0$ on $\mathbb{P}^2$.

Combined with irreducibility of $S(d)$ and $\dim S(d) = \dim \mathrm{Fol}(d)$:

Corollary
If $\mathcal{C}_k(d) = \mathrm{Fol}(d)$, then through every singularity of every $\mathcal{F} \in \mathrm{Fol}(d)$ passes an $\mathcal{F}$-invariant curve of degree k.

So it suffices to exhibit, for each $d \geq 2$, a foliation with a singularity admitting no algebraic separatrix.

A singularity with no algebraic separatrix

Pick $\lambda, \mu, \gamma \in \mathbb{C}^*$ that are $\mathbb{Z}$-linearly independent, and let

$$\Omega = x^{d-1}yz(x+y+z)\left(\lambda\frac{dx}{x} + \mu\frac{dy}{y} + \gamma\frac{dz}{z} - (\lambda+\mu+\gamma)\frac{d(x+y+z)}{x+y+z}\right).$$

The associated $\mathcal{F} \in \mathrm{Fol}(d)$ has multi-valued first integral

$$F(x,y,z) = \frac{x^{\lambda}\, y^{\mu}\, z^{\gamma}}{(x+y+z)^{\lambda+\mu+\gamma}}$$

and an isolated singularity at $p = [\lambda : \mu : \gamma]$, off the four invariant lines.

Claim. No algebraic curve invariant by $\mathcal{F}$ passes through p.

Why. Such a curve would meet, say, $\{z = 0\}$ at a singularity q of $\mathcal{F}$, hence at $[1 : 0 : 0]$ or $[0 : 1 : 0]$. WLOG $q = [1 : 0 : 0]$. Near q, $F = u(x,y)\, x^{\lambda} y^{\mu}$ with u a holomorphic unit and $\lambda/\mu \notin \mathbb{Q}$; the only holomorphic separatrices through q are $\{x = 0\}$ and $\{y = 0\}$, neither passing through p. Contradiction.

Hence $\mathcal{C}_k(d) \neq \mathrm{Fol}(d)$ for all k, and Baire concludes. $\square$

Summary

- Residue divisor of a closed meromorphic 1-form, and Weil's residue theorem $c(\mathrm{Res}\,\omega) = 0$.
- Residue theorem for flat meromorphic connections on line bundles: $c(L) = -c(\mathrm{Res}\,\nabla)$.
- Bott's connection on $\mathcal{N}^*_{\mathcal{F}}|_Y$ along a smooth $\mathcal{F}$-invariant hypersurface Y.
- Camacho–Sad index theorem: $c(\mathcal{N}_Y) = \sum_S \mathrm{CS}(\mathcal{F}, Y, S)\, c(S)$.
- Very general foliation of degree $d \geq 2$ on $\mathbb{P}^2$ has no algebraic leaf.

In Lecture 2 we will discuss the existence of separatrices for codimension-one foliations on surfaces.

Codimension one foliations on projective manifolds

Lecture 2 — Existence of separatrices on surfaces

Jorge Vitório Pereira
IMPA

Ann Arbor, May 2026

Simons Collaboration on Moduli, Ann Arbor

Where we were

Yesterday we discussed:

- Codimension-one foliations $\mathcal{F}$ and their conormal bundle $\mathcal{N}^*_{\mathcal{F}}$.
- Closed meromorphic 1-forms; residue divisor; Weil's residue theorem.
- Bott's connection on $\mathcal{N}^*_{\mathcal{F}}|_Y$ along a smooth invariant hypersurface Y.
- Camacho–Sad index theorem $c(\mathcal{N}_Y) = \sum_S \mathrm{CS}(\mathcal{F}, Y, S)\, c(S)$.

Today we apply these tools to a foundational question: do germs of codimension-one foliations have invariant hypersurfaces through their singular points?

Today

- Separatrices: definition, and the existence question.
- Briot–Bouquet: separatrices for reduced singularities of dimension two.
- Simple singularities and Seidenberg's reduction.
- The Camacho–Sad existence theorem on surfaces.
- Proof on the dual graph of the exceptional divisor.
- Higher-dimensional analogues: Cano–Cerveau, Cano–Mattei.
- Failure in the dicritical case; Jouanolou and Gomez-Mont–Luengo.
- Recent work: Spicer–Svaldi.

Separatrices and simple singularities

$\mathcal{F}$ a germ of codimension-one foliation on $(\mathbb{C}^n, 0)$, with $0 \in \operatorname{sing}(\mathcal{F})$.

Definition
A separatrix of $\mathcal{F}$ through 0 is a germ of irreducible hypersurface $S \subset (\mathbb{C}^n, 0)$ such that $S \setminus \operatorname{sing}(\mathcal{F})$ is contained in a leaf of $\mathcal{F}|_{(\mathbb{C}^n,0)\setminus \operatorname{sing}(\mathcal{F})}$.

Equivalently: if ω generates $\mathcal{N}^*_{\mathcal{F}}$ near 0 and $S = \{f = 0\}$ is smooth and irreducible, then $\omega = h\, df + f\, \eta$ for some germ h and some holomorphic 1-form η. Dropping smoothness of S, h and η have to be allowed to be meromorphic.

In dimension two, a separatrix is just a germ of irreducible curve invariant by $\mathcal{F}$.

Why separatrices?

A foliation that admits a separatrix at every singular point is much closer to being globally tractable.

- Invariant hypersurfaces support Bott's connection and Camacho–Sad indices.
- Reduction of singularities of $\mathcal{F}$ is controlled by the local separatrices.

The existence of separatrices is far from obvious. We will see that it holds in dimension two, smooth ambient space, but fails in general.

Definition
A germ $\mathcal{F}$ of codimension-one foliation on $(\mathbb{C}^2, 0)$ has a simple (or reduced) singularity at 0 if it is defined by a holomorphic 1-form

$$\omega = \lambda_1\, x\, dy - \lambda_2\, y\, dx + (\text{higher order})$$

with $(\lambda_1, \lambda_2) \neq (0,0)$ and, if both $\lambda_i \neq 0$, $\lambda_1/\lambda_2 \notin \mathbb{Q}_{>0}$.

Two cases:

- Both $\lambda_i \neq 0$: non-degenerate simple singularity. Linear part of the dual vector field is invertible.
- Exactly one $\lambda_i = 0$: saddle-node. Linear part has a single non-zero eigenvalue.

Briot–Bouquet (1856)

Theorem
Let $\mathcal{F}$ be a germ of foliation on $(\mathbb{C}^2, 0)$ with a simple singularity at 0.

- If the singularity is non-degenerate, $\mathcal{F}$ has exactly two separatrices, transverse to each other, tangent to the eigendirections.
- If the singularity is a saddle-node, $\mathcal{F}$ has at least one separatrix (the strong separatrix), tangent to the eigendirection of the non-zero eigenvalue, with $\mathrm{CS}(\mathcal{F}, \text{strong sep}) = 0$.

Seidenberg's reduction of singularities

Theorem (Seidenberg, 1968)
Let $\mathcal{F}$ be a germ of foliation on $(\mathbb{C}^2, 0)$. There exists a finite composition of point blow-ups

$$\pi : (X, E) \longrightarrow (\mathbb{C}^2, 0)$$

such that every singularity of $\pi^*\mathcal{F}$ on $E = \pi^{-1}(0)$ is simple.

The exceptional divisor E is a tree of rational curves. After Seidenberg, the question of existence of separatrices reduces to a question on this tree.

The Camacho–Sad existence theorem

Camacho–Sad (1982)

Theorem (Camacho–Sad)
Every germ of codimension-one holomorphic foliation $\mathcal{F}$ on $(\mathbb{C}^2, 0)$ admits a separatrix.

The result is non-trivial: there are germs of foliations on $(\mathbb{C}^2, 0)$ whose linear part has a single zero eigenvalue, or is identically zero, and for which the separatrix is not visible at first order.

Several proofs are known. We will follow the strategy of de Almeida dos Santos (2020), which is conceptually transparent and adapts well to refinements.

Setup for the proof

Let $\pi : (X, E) \to (\mathbb{C}^2, 0)$ be a reduction of singularities by Seidenberg.

Two possibilities:

1. Some irreducible component of E is not invariant by $\pi^*\mathcal{F}$.
2. Every irreducible component of E is invariant by $\pi^*\mathcal{F}$.

Case (1) is trivial: a non-invariant exceptional component is transverse to $\pi^*\mathcal{F}$ at infinitely many points, each producing a germ of curve invariant by $\pi^*\mathcal{F}$, hence a separatrix of $\mathcal{F}$ via π.

The substance of the theorem lies in Case (2).

Strategy in Case (2)

Assume all components of E are invariant. We will reach a contradiction with the hypothesis that no singularity on E provides a separatrix transverse to E.

By Briot–Bouquet, every non-degenerate simple singularity on E produces two separatrices. If neither is transverse to E, both are contained in E.

A saddle-node always has a strong separatrix; if it points outside E, we are done. So we may assume every singularity is either a corner of E (two transverse E-components meeting) or a saddle-node whose strong separatrix is contained in E.

The dual graph Γ

Let Γ be the dual graph of E:

- vertices $\leftrightarrow$ irreducible components C_v of E,
- edges $\leftrightarrow$ intersection points of two components.

Γ is a tree of rational curves.

Pass to a connected subgraph $\Gamma_0 \subset \Gamma$ obtained by deleting the edges that correspond to saddle-nodes whose strong separatrix sits on the opposite side. After this surgery, every edge of Γ_0 corresponds to a non-degenerate simple singularity.

A cocycle from Camacho–Sad indices

For each edge CD in Γ_0 joining components C and D at $p = C \cap D$, define

$$\sigma_{CD} \;=\; -\,\mathrm{CS}(\pi^*\mathcal{F},\; C,\; p) \;\in\; \mathbb{C}^*.$$

At a non-degenerate simple singularity with eigenvalue ratio λ, a direct calculation gives $\mathrm{CS}(\pi^*\mathcal{F}, C, p) \cdot \mathrm{CS}(\pi^*\mathcal{F}, D, p) = 1$, hence

$$\sigma_{CD} \;=\; \sigma_{DC}^{-1}.$$

So $\sigma \in Z^1(\Gamma_0, \mathbb{C}^*)$.

Trivialising the cocycle

Γ_0 is a subtree of a tree, hence a tree. Therefore

$$H^1(\Gamma_0, \mathbb{C}^*) = 0.$$

There exist non-zero complex numbers $\{\lambda_C\}_{C \in \Gamma_0}$ such that for every edge CD,

$$\sigma_{CD} = \frac{\lambda_C}{\lambda_D}.$$

Form the $\mathbb{C}$-divisor

$$R = \sum_{C \in \Gamma_0} \lambda_C \, C \in \mathrm{Div}(X) \otimes \mathbb{C}.$$

The contradiction

Apply Camacho–Sad's identity on each $D \in \Gamma_0$:

$$R \cdot D \;=\; \lambda_D \left(D^2 + \sum_{C \neq D} \frac{\lambda_C}{\lambda_D} \, C \cdot D \right) \;=$$

$$=\; \lambda_D \left(D^2 - \sum_{p \in D} \mathrm{CS}(\pi^* \mathcal{F}, D, p) \right) \;=\; 0.$$

So R is orthogonal to every $D \in \Gamma_0$ for the intersection form.

But the intersection matrix of $E_0 = \sum_{C \in \Gamma_0} C$ is negative definite (it is contracted by π).

Hence $R = 0$, contradicting $\lambda_C \neq 0$. $\square$

Remarks on the proof

- Only ingredients used: Seidenberg's reduction; Briot–Bouquet for simple singularities; the Camacho–Sad index identity on each $C \subset E$; negative-definiteness of the exceptional intersection matrix; vanishing of H^1 of a tree.
- No assumption that $\mathcal{F}$ is non-dicritical: dicriticality of $\mathcal{F}$ corresponds precisely to Case (1) of the dichotomy.
- The same argument adapts to singular surfaces with contractible resolution dual graph (Camacho 1988; further refinements by Toma, J. Cano, Sebastiani, Ortiz-Bobadilla–Rosales-Gonzalez–Voronin, de Almeida dos Santos).
- False for arbitrary normal surfaces. Simplest counter-example = natural foliation at a neighborhood of a cusp of a Hilbert modular surface.

Higher dimensions

What about $\dim X \geq 3$?

The naïve extension fails.

Main Obstruction : dicritical foliations. A foliation on a complex manifold of dimension ≥ 3 may have no separatrix at all.

Jouanolou's theorem

Recall from last lecture:

Theorem (Jouanolou)
A very general codimension-one foliation on $\mathbb{P}^2$ of degree $d \geq 2$ has no algebraic invariant curve.

Equivalently, viewing the foliation as a homogeneous 1-form ω on $\mathbb{C}^3$ and taking the cone, a very general germ of codimension-one foliation on $(\mathbb{C}^3, 0)$ has no separatrix.

This shows that the dimension-two argument cannot be lifted naïvely: the cone construction takes Jouanolou's foliations to dicritical foliations on $(\mathbb{C}^3, 0)$.

The relevant hypothesis to impose is therefore non-dicriticality: the resolution should have only $\mathcal{F}$-invariant exceptional divisors.

Cano–Cerveau and Cano–Mattei

Theorem (Cano–Cerveau 1992; Cano–Mattei 1992)
Let $\mathcal{F}$ be a germ of non-dicritical codimension-one foliation on $(\mathbb{C}^n, 0)$, $n \geq 3$. Then $\mathcal{F}$ admits a separatrix.

In dimension three, Cano–Cerveau prove existence of a separatrix and also a refinement: if $\mathcal{F}$ is non-dicritical, ω generates $\mathcal{N}^*_{\mathcal{F}}$ with codimension-≥ 2 zero set, and γ is a non-trivial germ of curve through 0 with $\gamma^*\omega \equiv 0$, then there is a unique germ of separatrix containing γ.

Cano–Mattei extend this to all dimensions ≥ 3.

Foliations by curves

Theorem (Gomez-Mont–Luengo, 1992)
There exist explicit germs of holomorphic foliations of dimension one on $(\mathbb{C}^3, 0)$ without invariant curves (formal or convergent) through zero.

Reduction of foliation singularities

The proof of Cano–Cerveau and Cano–Mattei rests on the reduction of singularities of codimension-one foliations.

Theorem (Cano, 2004)
Let $\mathcal{F}$ be a codimension-one foliation on a complex manifold X of dimension three. There exists a finite composition of blow-ups along smooth centres $\pi : Y \to X$ such that every singularity of $\pi^*\mathcal{F}$ is simple in the sense of Cano–Cerveau.

In dimensions ≥ 4 the reduction of singularities is open. This is a fundamental gap in the theory.

Simple singularities in higher dimensions

Definition (Cano–Cerveau)
A germ $\mathcal{F}$ on $(\mathbb{C}^n, 0)$ has a simple singularity if there exist formal coordinates $x_1, \ldots, x_n$ and an integer $2 \leq r \leq n$ (the dimension type) such that $\mathcal{F}$ is defined by a closed meromorphic 1-form with no base locus, simple normal crossing polar divisor and no resonances.

After bimeromorphic modification, every closed meromorphic 1-form admits an expression of this kind near each point of its polar set (cf. Lecture 1).

Recent work: Spicer–Svaldi

Theorem (Spicer–Svaldi)
Let $\mathcal{F}$ be a germ of codimension-one foliation on $(\mathbb{C}^3, 0)$ with log canonical singularities. Then $\mathcal{F}$ admits a separatrix.

Log canonical singularities are defined in terms of the discrepancies of $K_{\mathcal{F}}$ under birational pull-back, mirroring the MMP definitions for varieties.

Note that there exist dicritical foliations with log canonical singularities, so this is not a corollary of Cano–Cerveau.

Looking ahead

Preview of Lecture 3

- Pull-back foliations $\mathcal{F} = \pi^*\mathcal{G}$ for $\pi : X \dashrightarrow Y$ a rational map onto a surface Y.
- Riccati foliations and Transversely projective foliations.
- The Cerveau–Lins Neto conjecture: every codimension-one foliation on a projective manifold is accounted for by one of the above families.
- Positive results toward the Cerveau–Lins Neto conjecture.

Summary

- Briot–Bouquet: simple singularities in $\mathbb{C}^2$ admit separatrices.
- Seidenberg: any germ on $(\mathbb{C}^2, 0)$ reduces to simple singularities via blow-ups.
- Camacho–Sad: combining the two, every germ on $(\mathbb{C}^2, 0)$ has a separatrix.
- Dimension ≥ 3: existence holds in the non-dicritical codim-one case (Cano–Cerveau, Cano–Mattei); fails for general dicritical codim-one germs (Jouanolou) and for foliations of higher codim (Gomez-Mont–Luengo); holds for log canonical singularities in dimension three (Spicer–Svaldi).
- Reduction of singularities of foliations in dimension ≥ 4 is open.

Codimension one foliations on projective manifolds

Lecture 3 — Pull-back and transversely projective foliations

Jorge Vitório Pereira
IMPA

Ann Arbor, June 2026

Simons Collaboration on Moduli, Ann Arbor

Where we were

- L1: closed meromorphic 1-forms, Bott's connection, Camacho–Sad; Jouanolou on absence of algebraic leaves.
- L2: existence of separatrices on surfaces; partial results in higher dimensions.

The organising question for Lectures 3 and 4 is the Cerveau–Lins Neto conjecture.

The Cerveau–Lins Neto conjecture

Conjecture (Cerveau–Lins Neto)
Let $\mathcal{F}$ be a codimension-one singular holomorphic foliation on a projective manifold X. If $\mathcal{F}$ is not the pull-back, under a rational map, of a foliation on a projective surface, then $\mathcal{F}$ is a transversely projective foliation.

In other words: the conjecture predicts that the only codimension-one foliations on projective manifolds are

- rational pull-backs of foliations on surfaces, and
- transversely projective foliations,

and these two classes overlap but together exhaust everything.

Today

- Algebraically integrable foliations; the Darboux–Jouanolou criterion and its proof.
- Pull-back foliations under rational maps; quasi-invariant hypersurfaces.
- $\mathbb{P}^1$-bundles, Riccati foliations, transversely projective foliations.
- Transversely affine and virtually transversely Euclidean structures.
- Corlette–Simpson and its quasi-projective extension.
- Structure theorem of transversely projective foliations, and a sketch of one ingredient.

Algebraically integrable foliations

Algebraically integrable foliations

Definition
A codimension-one foliation $\mathcal{F}$ on a projective manifold X is algebraically integrable if its leaves are algebraic.

For a codimension-one $\mathcal{F}$ the property is equivalent to either of the following.

- $\mathcal{F}$ admits a non-constant rational first integral $f \in \mathbb{C}(X)$: any local generator ω of $\mathcal{N}^*_{\mathcal{F}}$ satisfies $\omega \wedge df = 0$.
- There exists a dominant rational map $\pi : X \dashrightarrow C$ to a smooth curve, with $\mathcal{F}$ the pull-back of the foliation by points on C.

The algebraic leaves are the (closures of) irreducible components of the fibres of π.

Examples

Logarithmic with rational residues. If X has no holomorphic 1-forms logarithmic 1-forms (closed rational 1-forms with simple poles) with rational residues

$$\omega = \sum_{i=1}^{r} \lambda_i \frac{dh_i}{h_i}, \qquad \lambda_i \in \mathbb{Q},$$

defines an algebraically integrable foliation with rational first integral $\prod h_i^{\lambda_i}$ (after clearing denominators).

Pencils. For homogeneous f, g of the same degree on $\mathbb{P}^n$, the 1-form $\omega = g\,df - f\,dg$ has rational first integral f/g, defining the pencil $\{f = tg\}_{t\in\mathbb{P}^1}$.

Darboux (1878)

The starting point of the subject. Darboux considered polynomial 1-forms $\omega = A\,dx + B\,dy$ on $\mathbb{C}^2$ with $\deg \omega = d$, and *algebraic solutions*: curves $\{f_i = 0\}$ with $\omega \wedge df_i$ divisible by f_i.

Theorem (Darboux–Jouanolou)
If ω admits more than $\binom{d+1}{2}$ irreducible algebraic solutions, then ω has a rational first integral; that is, $\omega = h(g\,df - f\,dg)/(\text{something})$ for some rational f/g.

Darboux's argument was elementary: each algebraic solution produces a logarithmic 1-form (essentially df_i/f_i). Beyond a critical number, these forms become linearly dependent, and the dependency relation supplies a rational function constant on the leaves.

Jouanolou's theorem is the global, projective extension of this argument.

The Darboux–Jouanolou criterion

For a codimension-one foliation $\mathcal{F}$ on a projective manifold X, set

$$\ell \;=\; \dim NS(X) \;+\; h^0(X, \Omega^1_{\mathcal{F}}) \;-\; h^0(X, \Omega^1_X).$$

Let k denote the number of irreducible $\mathcal{F}$-invariant hypersurfaces.

Theorem (Jouanolou 1979; refined version, P.–Spicer 2020)
With the notation above,

1. if $k \geq \ell$, then $\mathcal{F}$ is transversely affine;
2. if $k \geq \ell + 1$, then $\mathcal{F}$ is defined by a closed logarithmic 1-form;
3. if $k \geq \ell + 2$, then $\mathcal{F}$ is algebraically integrable.

In particular: a codimension-one foliation on a projective manifold with infinitely many invariant hypersurfaces is algebraically integrable. This is the classical Darboux–Jouanolou theorem.

Step I. log 1-forms from invariant hypersurfaces

Write $\mathrm{Inv}(\mathcal{F})$ for the subgroup of $\mathrm{Div}(X)$ generated by the irreducible $\mathcal{F}$-invariant hypersurfaces, and let

$$c : \mathrm{Inv}(\mathcal{F}) \otimes \mathbb{C} \longrightarrow H^2(X, \mathbb{C})$$

be the Chern class map. Set $\mathrm{Inv}_0(\mathcal{F}) \otimes \mathbb{C} = \ker c$.

Hodge theory. For each $D \in \mathrm{Inv}_0(\mathcal{F}) \otimes \mathbb{C}$ there exists a closed logarithmic 1-form η_D on X with $\mathrm{Res}(\eta_D) = D$ (Weil's residue theorem). The form η_D is defined modulo $H^0(X, \Omega^1_X)$.

Invariance. Since D is supported on $\mathcal{F}$-invariant hypersurfaces, the restriction

$$\eta_D|_{\mathcal{T}_{\mathcal{F}}} \ \in \ H^0(X, \Omega^1_{\mathcal{F}})$$

is well-defined (contractions of η_D with vectors tangent to $\mathcal{F}$ are holomorphic).

Step II. counting forces a first integral

We have a linear map

$$\rho : \mathrm{Inv}_0(\mathcal{F}) \otimes \mathbb{C} \longrightarrow H^0(X, \Omega^1_{\mathcal{F}})/H^0(X, \Omega^1_X), \qquad D \longmapsto \eta_D|_{\mathcal{T}_{\mathcal{F}}}.$$

Dimensions.

$$\dim(\mathrm{Inv}_0(\mathcal{F}) \otimes \mathbb{C}) \geq k - \dim NS(X), \qquad \dim \mathrm{target} = h^0(\Omega^1_{\mathcal{F}}) - h^0(\Omega^1_X).$$

Step (3): algebraic integrability. If $k \geq \ell + 2$, then $\dim \ker \rho \geq 2$. Two linearly independent closed logarithmic 1-forms η, η' vanishing on $\mathcal{T}_{\mathcal{F}}$ are proportional: $\eta = f\,\eta'$ for some $f \in \mathbb{C}(X)$. Differentiating,

$$0 = d\eta = df \wedge \eta' + f\,d\eta' = df \wedge \eta'.$$

So $df \wedge \eta' = 0$, and f is a non-constant rational first integral of $\mathcal{F}$. $\square$

The intermediate statements

Closed logarithmic 1-form. For $k \geq \ell + 1$ the kernel of ρ is non-zero. A non-trivial η with $\eta|_{\mathcal{T}_{\mathcal{F}}} = 0$ vanishes on the tangent sheaf of $\mathcal{F}$, so it is a 1-form defining $\mathcal{F}$. It is closed and logarithmic with poles on (invariant) hypersurfaces.

Transversely affine. For $k \geq \ell$ one uses a parallel argument with the bundle $\mathcal{N}_{\mathcal{F}}$ in place of $\Omega^1_{\mathcal{F}}$. Each fibre of c over $c(\mathcal{N}_{\mathcal{F}})$ in $\mathrm{Inv}(\mathcal{F}) \otimes \mathbb{C}$ provides a flat logarithmic connection on $\mathcal{N}_{\mathcal{F}}$; for k large enough one of them restricts to Bott's partial connection along $\mathcal{T}_{\mathcal{F}}$, which is the definition of transversely affine.

Sharpness. Simple examples on $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ show that each of the three bounds is sharp.

Pull-back foliations

Pull-back of a foliation on a surface

Let Y be a projective surface, $\mathcal{G}$ a codimension-one foliation on Y, and $\pi : X \dashrightarrow Y$ a dominant rational map from a projective manifold X.

The pull-back $\mathcal{F} = \pi^*\mathcal{G}$ is the codimension-one foliation on X whose conormal sheaf is the saturation of $\pi^*\mathcal{N}^*_{\mathcal{G}}$; equivalently, a local generator of $\mathcal{N}^*_{\mathcal{F}}$ is the saturation of $\pi^*\omega_{\mathcal{G}}$ for any local generator $\omega_{\mathcal{G}}$ of $\mathcal{N}^*_{\mathcal{G}}$.

The fibres of π form a (codimension-two) subfoliation by algebraic leaves of $\mathcal{F}$.

Recognising pull-backs. Conversely, a codimension-one $\mathcal{F}$ on X is a pull-back from a surface if, and only if, it admits a codimension-two subfoliation by algebraic leaves.

Quasi-invariant hypersurfaces

Definition
An irreducible hypersurface $H \subset X$ is quasi-invariant by $\mathcal{F}$ if H is not $\mathcal{F}$-invariant and $\mathcal{F}|_H$ is algebraically integrable.

- Invariant hypersurfaces force $\mathcal{F}$ itself to be algebraically integrable when there are many (Darboux–Jouanolou).
- Pull-backs from surfaces $\pi : X \dashrightarrow Y$ produce a hypersurface for every curve on Y; the non-$\mathcal{G}$-invariant curves give quasi-invariant hypersurfaces on X, foliated by fibres of π.

Proposition (P.–Spicer)
If $\mathcal{F} = \pi^*\mathcal{G}$ with $\pi : X \dashrightarrow Y$ dominant to a surface, then $\pi^* \operatorname{Div}(Y) \subset \operatorname{Inv}(\mathcal{F}) + \operatorname{QInv}(\mathcal{F})$.

Quasi-invariant hypersurfaces force a pull-back

Theorem (P.–Spicer (2020))
Let $\mathcal{F}$ be a codimension-one holomorphic foliation on a projective manifold X. If $\mathcal{F}$ admits infinitely many quasi-invariant hypersurfaces, then either

1. $\mathcal{F}$ is algebraically integrable, or
2. $\mathcal{F}$ is a pull-back of a foliation on a projective surface under a dominant rational map.

Explicit bound for non-dicritical. If $\mathcal{F}$ has non-dicritical singularities on a 3-dimensional general complete intersection, then

$$\dim \mathrm{QInv}(\mathcal{F}) \otimes \mathbb{C} \;\geq\; \dim \frac{NS(X) \otimes \mathbb{C}}{c(\mathrm{Inv}(\mathcal{F}) \otimes \mathbb{C})} \;+\; 2$$

already forces $\mathcal{F}$ to be algebraically integrable or a pull-back from a surface.

A direct analogue of Darboux–Jouanolou, with quasi-invariant in place of invariant.

Sketch I — reduction and S^1-flat divisors

Reduce. A general 3-dimensional complete intersection $V \subset X$ inherits the same Inv/QInv data (Lefschetz); so one reduces to $\dim X = 3$. A further resolution makes $\mathcal{F}$ non-dicritical (Cano 2004), under which $\mathrm{Inv}(\mathcal{F}) + \mathrm{QInv}(\mathcal{F})$ pulls back well.

S^1-flat divisors. A divisor $D \in \mathrm{Div}(X) \otimes \mathbb{C}$ is S^1-flat if its line bundle is in the image of $H^1(X, S^1) \to \mathrm{Pic}(X)$. On a compact Kähler manifold this is equivalent to $c(D) = 0$ in $H^2(X, \mathbb{C})$.

Associated 1-form. An S^1-flat D comes with a closed logarithmic 1-form ω_D with purely imaginary periods, and a continuous real first integral $F = |\exp \int \omega_D| : X \to [0, \infty]$ of the foliation $\mathcal{F}_D$ defined by ω_D.

The hypothesis produces two S^1-flat divisors $D_1, D_2 \in \mathrm{Inv}(\mathcal{F}) + \mathrm{QInv}(\mathcal{F})$ with disjoint quasi-invariant components.

Sketch II — restriction to a general leaf

After blowing up, write $D_i = D_{i,+} - D_{i,-}$ with $D_{i,+}$ and $D_{i,-}$ disjoint effective.

Let $H_1 \subset \mathrm{supp}(D_1)$ be a quasi-invariant hypersurface and $j : L \to X$ a general leaf of $\mathcal{F}$ passing through a general point of H_1. Let E be the connected component of $j^*(D_1)$ containing a component of j^*H_1.

Properties of E.

- E is effective on L (or $-E$ is, by symmetry).
- $c(j^*D_1) = 0$ in $H^2(L, \mathbb{C})$, so $E^2 = 0$.
- Every curve $C \subset \mathrm{supp}(E)$ has $E \cdot C = 0$: E is nef.

The dichotomy now depends on how E meets $j^*(D_2)$.

Sketch III — algebraicity vs. deformation

Case A: $\mathrm{supp}(j^*D_2) \cap \mathrm{supp}(E) \neq \emptyset$. Pick a curve $F \subset \mathrm{supp}(j^*D_2)$ with $E \cdot F > 0$. For $k \gg 0$,

$$(kE + F)^2 \;=\; k^2E^2 + 2k\,E \cdot F + F^2 \;>\; 0,$$

and $kE + F$ is effective and nef. By a Bogomolov-type criterion on the closure of a leaf in a 3-fold, L has algebraic Zariski closure. Generality of L gives $\mathcal{F}$ algebraically integrable.

Case B: $\mathrm{supp}(j^*D_2) \cap \mathrm{supp}(E) = \emptyset$. The first integral of $\mathcal{F}_{D_1}|_L$ takes a real interval of values on L; a maximum principle argument (lifting the multi-valued function $\exp\int \omega_{D_1}$) shows a saturated neighbourhood of E is foliated by compact $\mathcal{F}_{D_1}$-invariant curves. So E moves in a one-parameter family of compact curves on L, which assembles into a foliation by algebraic curves tangent to $\mathcal{F}$. This subfoliation gives a dominant rational map $X \dashrightarrow Y$ to a surface with $\mathcal{F} = \pi^*\mathcal{G}$. $\square$

An application

Cone of curves. On a $\mathbb{Q}$-factorial projective 3-fold with klt singularities, a codim-one foliation with non-dicritical canonical singularities whose $K_{\mathcal{F}}$-negative cone has infinitely many extremal rays is either algebraically integrable with rational general leaf, or all but finitely many extremal rays sit on finitely many rational $\mathcal{F}$-invariant hypersurfaces (P.–Spicer, Theorem C).

Transversely projective foliations

Riccati foliations

Let $\pi : E \to X$ be a $\mathbb{P}^1$-bundle on a complex manifold X.

Definition
A Riccati foliation on E is a codimension-one foliation transverse to a general fibre of π.

Examples are produced by the Riemann–Hilbert correspondence: a representation $\rho : \pi_1(X \setminus D) \to \mathrm{PGL}_2(\mathbb{C})$ gives a $\mathbb{P}^1$-bundle over X with a flat connection on $X \setminus D$, and integrating the connection gives a Riccati foliation with poles along D.

The leaf space of a Riccati foliation is locally $\mathbb{P}^1$, hence transverse models for the foliation are open subsets of $\mathbb{P}^1$ with $\mathrm{PGL}_2(\mathbb{C})$-transition data.

Transversely projective foliations

Definition
A codimension-one foliation $\mathcal{F}$ on a projective manifold X is transversely projective if there exists a $\mathbb{P}^1$-bundle $\pi : E \to X$, a Riccati foliation $\mathcal{R}$ on E, and a rational section $\sigma : X \dashrightarrow E$ such that $\mathcal{F} = \sigma^* \mathcal{R}$.

In terms of 1-forms: $\mathcal{F}$ is defined by a rational 1-form ω_0 such that there exist rational 1-forms ω_1, ω_2 with

$$d\omega_0 = \omega_0 \wedge \omega_1, \quad d\omega_1 = \omega_0 \wedge \omega_2, \quad d\omega_2 = \omega_1 \wedge \omega_2.$$

The triple $(\omega_0, \omega_1, \omega_2)$ encodes a transverse $\mathfrak{sl}_2(\mathbb{C})$-structure.

Three transverse Lie algebras

Specialising the triple gives three nested classes.

- Transversely projective: full triple, transverse Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.
- Transversely affine: $\omega_2 = 0$, so $d\omega_0 = \omega_0 \wedge \omega_1$ and $d\omega_1 = 0$. Transverse Lie algebra $\mathfrak{aff}(\mathbb{C})$.
- Virtually transversely Euclidean: ω_1 closed logarithmic with periods in $\mathbb{Q} \cdot 2\pi i$. Transverse Lie algebra $\mathbb{C}$. Equivalently, $\mathcal{F}$ is defined by a closed rational 1-form after a finite ramified covering.

Algebraically integrable foliations sit inside virtually transversely Euclidean (any rational first integral f gives df, closed rational).

Examples

Closed rational 1-forms. If $\mathcal{F}$ is defined by $\omega \in \Omega^1(X)$ rational with $d\omega = 0$, then $\mathcal{F}$ is virtually transversely Euclidean.

Logarithmic foliations. $\omega = \sum \lambda_i \, dh_i/h_i$ is closed; $\mathcal{F}_\omega$ is transversely Euclidean. These form a Zariski-open subset of an irreducible component of $\mathrm{Fol}(\mathbb{P}^n)$.

Halphen / classical examples. Several classical foliations on $\mathbb{P}^2$ and $\mathbb{P}^3$ are transversely projective via $\mathrm{PGL}_2(\mathbb{C})$-representations of fundamental groups of complements of curves.

Riccati pull-backs. Any pull-back of a Riccati foliation on a ruled surface by a dominant rational map is transversely projective.

Structure results

Corlette–Simpson, and an extension

The structure theorem for transversely projective foliations rests on a classification of rank-two representations of fundamental groups.

Theorem (Corlette–Simpson, *Compositio* 144 (2008))
Let X be a smooth projective variety, D a simple normal-crossing divisor and $\rho : \pi_1(X - D) \to \mathrm{SL}_2(\mathbb{C})$ a Zariski-dense, quasi-unipotent at infinity. Then ρ projectively factors through

1. a morphism $X \to Y$ to an orbicurve, or
2. a morphism $X \to \mathcal{H}$ to a polydisk Shimura modular orbifold, equipped with one of its tautological $\mathrm{PSL}_2(\mathbb{C})$-representations.

A polydisk Shimura modular orbifold is an arithmetic quotient $\mathbb{D}^n/\Gamma$ of a product of discs, built from a quaternion algebra over a totally real field. Hilbert modular surfaces are the simplest examples; in higher arithmetic rank one gets projective examples.

Corlette–Simpson, quasi-projective version

Transversely projective foliations on X produce representations of $\pi_1(X \setminus D)$, which are not necessarily quasi-unipotent at infinity. So Corlette–Simpson is not enough.

Theorem (Loray–P.–Touzet, 2016)
Let X° be a quasi-projective manifold and $\rho : \pi_1(X^\circ) \to \mathrm{SL}_2(\mathbb{C})$ a Zariski-dense representation which is not quasi-unipotent at infinity. Then ρ projectively factors through a morphism $X^\circ \to Y$ to an orbicurve.

The complement is handled by Corlette–Simpson after passing to a suitable compactification.

Structure of transversely projective foliations

Theorem D, Loray–P.–Touzet (2016)
Let $\mathcal{F}$ be a codimension-one transversely projective foliation on a projective manifold X. Then at least one of the following holds.

1. There is a generically finite Galois cover $f : Y \to X$ such that $f^*\mathcal{F}$ is defined by a closed rational 1-form.
2. There is a rational dominant map $f : X \dashrightarrow S$ to a ruled surface $\pi : S \to C$, and a Riccati foliation $\mathcal{H}$ on S, with $\mathcal{F} = f^*\mathcal{H}$.
3. There is a polydisk Shimura modular orbifold $\mathcal{H}$, a tautological Riccati foliation $\mathcal{H}_\rho$ on $\mathcal{H} \times_\rho \mathbb{P}^1$, and a rational map $f : X \dashrightarrow \mathcal{H} \times_\rho \mathbb{P}^1$ with $\mathcal{F} = f^*\mathcal{H}_\rho$.

Case (1) is the virtually transversely Euclidean case; case (2) is what we have classically called pull-back from a Riccati foliation; case (3) is the genuinely new one, and it is unavoidable.

Remarks on the three cases

Case (3) is rare but real. After a sequence of blow-ups, foliations in case (3) are defined locally near their singular set by
$$\sum_i \lambda_i \frac{dx_i}{x_i}, \quad \lambda_i \in \mathbb{R}.$$
The associated representation $\pi_1^{\mathrm{orb}}(\mathcal{H}) \to \mathsf{PSL}_2(\mathbb{C})$ is Zariski dense, rigid, and quasi-unipotent at infinity.

The cases overlap. A foliation may be a pull-back from a ruled surface *and* virtually defined by a closed rational form; the theorem asserts existence of at least one description, not uniqueness.

Companion result. The same trichotomy holds for flat meromorphic $\mathfrak{sl}_2$-connections on X. Connections with non-trivial Stokes matrices land in case (2): irregular $\mathfrak{sl}_2$-connections projectively factor through a curve.

Holonomy of algebraic leaves

Theorem (Claudon–Loray–P.–Touzet (2019))
Let $\mathcal{F}$ be a codimension-one foliation on a projective manifold X and let $L \subset X$ be an algebraic leaf. The holonomy representation

$$\rho_L : \pi_1(L^\circ) \longrightarrow \mathrm{Diff}(\mathbb{C}, 0)$$

satisfies the following dichotomy: either it factors through a curve (so $\mathcal{F}$ is a pull-back from a surface), or its image lies in a solvable subgroup of $\mathrm{Diff}(\mathbb{C}, 0)$.

This is direct evidence for the Cerveau–Lins Neto conjecture: solvable holonomy is what one expects of a transversely projective foliation.

Sketch: factoring representations through curves

The non-quasi-unipotent case

We zoom in on the proof of the result: a Zariski-dense $\rho : \pi_1(X^\circ) \to \mathrm{SL}_2(\mathbb{C})$ that is not quasi-unipotent at infinity factors projectively through an orbicurve.

Strategy. The three ingredients are

- Residue theorem,
- Malcev's residual finiteness for finitely generated linear groups,
- Totaro's fibration criterion (three disjoint divisors with proportional Chern classes come from a map to a curve).

From non-quasi-unipotency to indefiniteness

Take $E = \bigcup E_i \subset D$ maximal such that $\rho(\gamma_i) \neq \pm\mathrm{Id}$ for each small loop γ_i around E_i and at least one γ_i is not quasi-unipotent.

On a tubular neighbourhood U of E, the projectivisation of $\rho|_{\pi_1(U\setminus E)}$ has solvable image (the non-quasi-unipotent loop produces a Borel-invariant line). Realise this solvable monodromy by a logarithmic flat connection on a line subbundle $F_1 \hookrightarrow F$ (Deligne's extension).

The residue formula gives

$$A \cdot v = b, \quad A = (E_i \cdot E_j), \quad 2b \in \mathbb{Z}^k.$$

If A were negative-definite, A would be invertible and v rational, forcing ρ to be quasi-unipotent on every E_i: contradiction. So A is indefinite.

From indefiniteness to a fibration

Malcev's residual finiteness produces a finite cover $\varphi : Y \to X$ ramified along D such that the preimage of E has at least three disjoint connected components.

Indefiniteness of A plus Hodge index theory upgrades these to three pairwise disjoint effective divisors F_1, F_2, F_3 on Y with $F_i \cdot F_i = 0$ and proportional Chern classes in $H^2(Y, \mathbb{R})$.

Totaro's criterion
Three pairwise disjoint connected effective divisors with Chern classes on a line in $H^2(Y, \mathbb{R})$ come from a morphism $g : Y \to \Sigma$ to a smooth curve.

Simplicity of $\mathrm{PSL}_2(\mathbb{C})$ forces $\rho|_{\pi_1(\text{fibre})}$ to be trivial. Galois descent of φ drops Σ to an orbicurve on X. $\square$

Status

Status of the conjecture

- **Known** on $\mathbb{P}^2$ trivially: any codim-one foliation on a surface is its own pull-back.
- **Known** for foliations with $K_{\mathcal{F}} \equiv 0$ and canonical singularities.
- **Known** for pencils on any projective manifold.
- **Known** for foliations with infinitely many quasi-invariant hypersurfaces.
- Strong evidence from holonomy representations.
- **Open** in general, even on $\mathbb{P}^3$.

Codimension one foliations on projective manifolds

Lecture 4 — Reduction to positive characteristic

Jorge Vitório Pereira
IMPA

Ann Arbor, June 2026

Simons Collaboration on Moduli, Ann Arbor

Where we were

- L1: closed meromorphic 1-forms; index theorems.
- L2: existence of separatrices on surfaces.
- L3: the Cerveau–Lins Neto conjecture; transversely projective foliations; structure results in characteristic zero.

Recall the conjecture

Conjecture (Cerveau–Lins Neto)
A codimension-one foliation $\mathcal{F}$ on a projective manifold X is either a rational pull-back of a foliation on a projective surface, or transversely projective.

Today

Reduction to positive characteristic.

- Foliations over a field of characteristic $p > 0$; p-closed and p-dense foliations.
- The p-curvature $\psi_{\mathcal{F}}$, the p-distribution $V(\mathcal{F})$, and the degeneracy divisor $\Delta_{\mathcal{F}}$.
- The Cartier operator and the Cartier transform.
- Spreading out and lifting structures back to characteristic zero.
- Applications.

Other results related to the conjecture.

- Deformation of rational curves and the transverse Lie algebra.
- Pencils of integrable 1-forms and unlikely intersections.

Foliations in characteristic p

Setup

k an algebraically closed field of characteristic $p > 0$. X a smooth projective variety over k.

A foliation $\mathcal{F}$ on X is, exactly as in characteristic zero, a pair $(\mathcal{T}_{\mathcal{F}}, \mathcal{N}^*_{\mathcal{F}}) \subset (T_X, \Omega^1_X)$ each annihilating the other, with $\mathcal{T}_{\mathcal{F}}$ involutive.

New phenomenon: for a derivation v on X/k, the p-fold composition

$$v^p : \mathcal{O}_X \to \mathcal{O}_X$$

satisfies Leibniz and is again a derivation. So T_X acquires an extra structure $v \mapsto v^p$ (p-power operation).

Two extremes: p-closed and p-dense

Definition
$\mathcal{F}$ is p-closed if $v \in \mathcal{T}_{\mathcal{F}} \Rightarrow v^p \in \mathcal{T}_{\mathcal{F}}$ for every local section v.

A p-closed codimension-q foliation has conormal generated (up to saturation) by $df_1, \ldots, df_q$: it has many rational first integrals, descended through Frobenius.

Definition
$\mathcal{F}$ is p-dense if $\mathcal{O}_{X/\mathcal{F}} = \mathcal{O}_{X^{(p)}}$. Every rational first integral is a p-th power.

The two notions are opposite: p-closed has lots of first integrals, p-dense has none beyond p-th powers. The interesting case is p-dense.

From p-dense to a closed rational 1-form

Proposition
Let X/k be smooth, ω an integrable rational 1-form on X, $\mathcal{F} = \mathcal{F}_\omega$. If there is a rational vector field ξ with

$$\iota_\xi \omega = 0 \quad \text{and} \quad F := \omega(\xi^p) \not\equiv 0,$$

then $F^{-1}\omega$ is closed.

More generally,

Proposition. A p-dense codim-q foliation is defined by closed rational 1-forms $\omega_1, \ldots, \omega_q$.

Proof sketch of the codimension-one case: setup

As in characteristic zero, $\mathcal{T}_{\mathcal{F}}$ is generated locally near a smooth point by commuting rational vector fields $v_1, \ldots, v_{n-1}$ with $\iota_{v_i}\omega = 0$ and $[v_i, v_j] = 0$.

Write $\xi = \sum_j a_j v_j$ for an arbitrary rational vector field tangent to $\mathcal{F}$. Jacobson's formula

$$(u + w)^p \;=\; u^p + w^p + Q_p(u, w), \qquad Q_p \in \mathrm{Lie}(u, w),$$

combined with $[v_i, v_j] = 0$, yields

$$\xi^p \;\equiv\; \zeta \;:=\; \sum_j a_j^p\, v_j^p \pmod{\langle v_1, \ldots, v_{n-1}\rangle}.$$

Since $\omega(v_j) = 0$, we have $\omega(\zeta) = \omega(\xi^p) = F$. Set $\alpha := F^{-1}\omega$; then $\alpha(\zeta) = 1$ and $\alpha(v_j) = 0$.

Proof sketch of the codimension-one case: Lie bracket computation

Key brackets. Since $[v_\ell, v_j] = 0$, an iterated bracket gives $[v_\ell^p, v_j] = \mathrm{ad}(v_\ell)^p(v_j) = 0$. In characteristic p, $v_j(a_\ell^p) = p\, a_\ell^{p-1}\, v_j(a_\ell) = 0$. Hence

$$[\zeta, v_j] \;=\; \sum_\ell \left(a_\ell^p\, [v_\ell^p, v_j] - v_j(a_\ell^p)\, v_\ell^p \right) \;=\; 0.$$

Integrability. $\omega \wedge d\omega = 0$ propagates to $\alpha \wedge d\alpha = 0$. Contracting with ζ and using $\alpha(\zeta) = 1$,

$$0 = \iota_\zeta(\alpha \wedge d\alpha) = d\alpha - \alpha \wedge \iota_\zeta d\alpha.$$

Apply Cartan formula. As $\iota_\zeta \alpha = 1$ is constant, $\mathcal{L}_\zeta \alpha = d(\iota_\zeta \alpha) + \iota_\zeta d\alpha = \iota_\zeta d\alpha$. On the frame $\{\zeta, v_1, \ldots, v_{n-1}\}$,

$$(\mathcal{L}_\zeta \alpha)(v_j) = \zeta(\alpha(v_j)) - \alpha([\zeta, v_j]) = 0, \qquad (\mathcal{L}_\zeta \alpha)(\zeta) = \zeta(1) = 0,$$

so $\mathcal{L}_\zeta \alpha = 0$. Therefore $\iota_\zeta d\alpha = 0$ and $d\alpha = \alpha \wedge \iota_\zeta d\alpha = 0$. $\square$

p-curvature, $V(\mathcal{F})$, and the degeneracy divisor

p-curvature of a foliation

The intrinsic invariant of the p-power operation on $\mathcal{F}$ is the p-curvature morphism:

$$\psi_{\mathcal{F}} : \mathrm{Frob}_* \mathcal{T}_{\mathcal{F}} \longrightarrow \mathcal{N}_{\mathcal{F}}, \qquad \sum f_i \otimes v_i \longmapsto \sum f_i\, v_i^p \mod \mathcal{T}_{\mathcal{F}}.$$

It is $\mathcal{O}_X$-linear (after Frobenius pushforward) and vanishes identically iff $\mathcal{F}$ is p-closed.

For a codimension-one $\mathcal{F}$, the image of $\psi_{\mathcal{F}}$ is an ideal multiple of $\mathcal{N}_{\mathcal{F}}$:

$$\mathrm{Im}\, \psi_{\mathcal{F}} = \mathcal{N}_{\mathcal{F}} \otimes \mathcal{I}_Z.$$

The p-distribution $V(\mathcal{F})$

Proposition. There exists a codimension two distribution $V(\mathcal{F})$ contained in $\mathcal{F}$ such that

$$\ker(\psi_{\mathcal{F}}) = \mathsf{Frob}_* \mathcal{T}_{V(\mathcal{F})} \subset \mathsf{Frob}_* \mathcal{T}_{\mathcal{F}}.$$

It is the saturated sub-distribution of $\mathcal{F}$ on which $v \mapsto v^p$ stays inside $\mathcal{T}_{\mathcal{F}}$.

Pull-back case. If $\mathcal{F} = \pi^*\mathcal{G}$ for a dominant rational $\pi : X \dashrightarrow Y$, the foliation by fibres of π has algebraic leaves, hence is preserved by $v \mapsto v^p$. It is contained in $V(\mathcal{F})$, and equals it generically.

The degeneracy divisor $\Delta_{\mathcal{F}}$

Definition
The divisorial part of $\mathcal{I}_Z$ is the degeneracy divisor of the p-curvature, $\Delta_{\mathcal{F}}$.

Proposition. If $\mathcal{F}$ is p-dense of codimension one,

$$\mathcal{O}_X(\Delta_{\mathcal{F}}) \cong \left(\omega_{\mathcal{F}} \otimes \omega^*_{V(\mathcal{F})}\right)^{\otimes p} \otimes \mathcal{N}_{\mathcal{F}}.$$

Proposition. Every $\mathcal{F}$-invariant hypersurface lies in $\operatorname{supp}(\Delta_{\mathcal{F}})$. Conversely, if $H \subset \operatorname{supp}(\Delta_{\mathcal{F}})$ with $p \nmid \operatorname{ord}_H(\Delta_{\mathcal{F}})$, then H is $\mathcal{F}$-invariant.

Proposition. For any closed rational η defining $\mathcal{F}$, the coefficients of $\Delta_{\mathcal{F}}$ and of $(\eta)_\infty - (\eta)_0$ agree modulo p.

The Cartier operator and the Cartier transform

The Cartier operator

On smooth X/k, Cartier proved the existence of a unique morphism of sheaves of graded-commutative $\mathcal{O}_X$-algebras

$$C \;:\; \mathrm{Frob}_* \, Z\Omega_X^\bullet \;\longrightarrow\; \Omega_X^\bullet$$

satisfying:

- $\ker C = \mathrm{Frob}_* \, B\Omega_X^\bullet$ (it kills exact forms);
- $C(f^{p-1}\, df) = df$ for every $f \in \mathcal{O}_X$;
- C is surjective, inducing $H^\bullet(\mathrm{Frob}_* \, \Omega_X^\bullet) \;\cong\; \Omega_X^\bullet$ as $\mathcal{O}_X$-algebras.

Domain. C acts on closed forms only. For a non-closed ω, the symbol $C(\omega)$ has no meaning.

Key identity. For $\omega \in Z\Omega_X^1$ and $v \in T_X$,

$$\big(\iota_v \, C(\omega)\big)^p \;=\; \iota_{v^p}\omega \;-\; v^{p-1}\big(\iota_v\omega\big).$$

The Cartier transform of a foliation

Definition
Let $\mathcal{F}$ be a p-dense codimension-q foliation on X. The Cartier transform $C(\mathcal{F})$ is the distribution

$$C(\mathcal{F}) = \bigcap_{\omega \in Z\Omega^1_X,\ \omega|_{T_{\mathcal{F}}}=0} \ker C(\omega) \subset T_X.$$

The intersection is well-defined: between two closed defining forms ω and ω' one has $\omega' = g^p \omega$ for some rational g, and $C(g^p\omega) = g\, C(\omega)$.

Caveat. $C(\mathcal{F})$ is not always integrable. On $\mathbb{A}^3$ take

$$\omega = y^{p-1}\, dy + z^p\, x^{p-1}\, dx, \qquad C(\omega) = dy + z\, dx,$$

whose kernel distribution is not involutive.

Theorem. If $p > 2$ and $\mathcal{F}$ lifts to $W_2(k)$, then $C(\mathcal{F})$ *is* a foliation.

$V(\mathcal{F})$ via the Cartier transform

Proposition. For a p-dense codim-one foliation $\mathcal{F}$ on X,

$$V(\mathcal{F}) \;=\; \overline{\mathcal{T}_{\mathcal{F}} \cap \mathcal{T}_{C(\mathcal{F})}} \quad \text{(saturation in } T_X\text{)}.$$

So $V(\mathcal{F})$ admits two descriptions:

- intrinsic, as the kernel of the p-curvature $\psi_{\mathcal{F}}$;
- via the Cartier transform, as the directions tangent to both $\mathcal{F}$ and $C(\mathcal{F})$.

Spreading out

From complex to mod p

A complex codim-one foliation $\mathcal{F}$ on a projective manifold X is defined by finitely many equations and 1-forms, all with finitely many coefficients.

Choose a finitely generated $\mathbb{Z}$-subalgebra $R \subset \mathbb{C}$ over which $X, \mathcal{F}$ are defined. For each maximal ideal $\mathfrak{p} \subset R$, $R/\mathfrak{p} = k(\mathfrak{p})$ is a finite field of characteristic $p > 0$.

Reduction modulo $\mathfrak{p}$. $X_{\mathfrak{p}} = X \otimes_R k(\mathfrak{p})$, $\mathcal{F}_{\mathfrak{p}}$ the corresponding foliation.

The arithmetic question: how often is $\mathcal{F}_{\mathfrak{p}}$ not p-closed?

If p-closed for almost every $\mathfrak{p}$, that is for an open subset of maximal primes, then, the Ekedahl–Shepherd-Barron–Taylor conjecture predicts $\mathcal{F}$ is algebraically integrable.

Spreading-out principle

Proposition
Let $\mathcal{F}$ be a foliation on a polarised projective manifold (X, H) defined over a finitely generated $\mathbb{Z}$-algebra $R \subset \mathbb{C}$. If there are integers M, m and a Zariski dense set of maximal primes $\mathfrak{P} \subset \mathrm{Spec}(R)$ such that $\mathcal{F}_{\mathfrak{p}}$ has an invariant subvariety of dimension m and degree $\leq M$ for every $\mathfrak{p} \in \mathfrak{P}$, then $\mathcal{F}$ itself has such an invariant subvariety.

The same principle applies to other bounded structures: closed rational 1-forms with polar divisor of bounded degree, transverse symmetries with bounded poles, and sub-foliations of bounded degree, **in particular** $V(\mathcal{F}_{\mathfrak{p}})$**.**

Application: numerically trivial canonical class

Proposition (Loray–P.–Touzet, 2018)
Let $\mathcal{F}$ be a codim-one foliation on a complex projective manifold X with $\omega_{\mathcal{F}}$ numerically trivial. If $\mathcal{F}_{\mathfrak{p}}$ is not p-closed for a Zariski dense set of primes, then there exist

- a line bundle L on X carrying a flat connection ∇, and
- a ∇-closed rational 1-form $\eta \in H^0\big(X, \Omega^1_X(*) \otimes L\big)$,

such that η defines $\mathcal{F}$.

Step 1: degree of the degeneracy divisor

Fix an ample class H on X. The formula

$$\mathcal{O}_X(\Delta_{\mathcal{F}_\mathfrak{p}}) \;\cong\; \left(\omega_{\mathcal{F}} \otimes \omega^*_{V(\mathcal{F}_\mathfrak{p})}\right)^{\otimes p} \otimes \mathcal{N}_{\mathcal{F}},$$

intersected with H^{n-1} and using $\omega_{\mathcal{F}} \cdot H^{n-1} = 0$, gives

$$(\Delta_{\mathcal{F}_\mathfrak{p}} + p\,\omega_{V(\mathcal{F}_\mathfrak{p})}) \cdot H^{n-1} \;=\; \mathcal{N}_{\mathcal{F}} \cdot H^{n-1}$$

Both summands on the right are non-negative. The divisor in question is effective and $\mathcal{T}_{\mathcal{F}}$ is semi-stable (Miyaoka).

Their sum is the fixed integer $\mathcal{N}_{\mathcal{F}} \cdot H^{n-1}$, and the second is a multiple of p. Hence for $p > \mathcal{N}_{\mathcal{F}} \cdot H^{n-1}$,

$$\omega_{V(\mathcal{F}_\mathfrak{p})} \cdot H^{n-1} = 0, \qquad \Delta_{\mathcal{F}_\mathfrak{p}} \cdot H^{n-1} = \mathcal{N}_{\mathcal{F}} \cdot H^{n-1}.$$

Step 2: closed defining form with flat coefficients

For $\mathfrak{p}$ in the dense set, $\mathcal{F}_\mathfrak{p}$ is defined by a closed rational 1-form $\eta_\mathfrak{p}$, unique up to $\eta_\mathfrak{p} \mapsto h^p \eta_\mathfrak{p}$. The coefficient comparison gives

$$\Delta_{\mathcal{F}_\mathfrak{p}} \;=\; (\eta_\mathfrak{p})_\infty - (\eta_\mathfrak{p})_0 + p\, D_\mathfrak{p},$$

for some integral divisor $D_\mathfrak{p}$. Since $\mathcal{N}_\mathcal{F} \cong \mathcal{O}((\eta_\mathfrak{p})_\infty - (\eta_\mathfrak{p})_0)$, intersecting with H^{n-1} and using Step 1 forces $D_\mathfrak{p} \cdot H^{n-1} = 0$. Adjusting $\eta_\mathfrak{p} \mapsto \eta_\mathfrak{p}/h_\mathfrak{p}^p$ absorbs the zeros into a p-th power, leaving

$$(\eta_\mathfrak{p})_0 = 0, \qquad (\eta_\mathfrak{p})_\infty = \Delta_{\mathcal{F}_\mathfrak{p}}.$$

The closed form $\eta_\mathfrak{p}$ is a regular section of $\Omega^1_{X_\mathfrak{p}} \otimes \mathcal{N}_\mathcal{F}$.

Lift. The spreading-out lemma for closed rational 1-forms of bounded polar degree produces $\eta \in H^0(X, \Omega^1_X(*) \otimes L)$ defining $\mathcal{F}$, with L of the same numerical class as $\mathcal{N}_\mathcal{F}$. $\square$

Components of $\mathrm{Fol}_d(\mathbb{P}^n)$

A family of new components

Theorem
For every $n \geq 3$ and every pair of integers $a \geq 2$, $b \geq 3$, the scheme $\mathrm{Fol}_{a+b-2}(\mathbb{P}^n_{\mathbb{C}})$ has an irreducible component whose general element is of the form $\mathcal{F} = \pi^*\mathcal{G}$, where

- $\mathcal{G}$ is a foliation on $\mathbb{P}^1 \times \mathbb{P}^1$ with $\mathcal{N}_{\mathcal{G}} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b)$;
- $\pi(x_0 : \cdots : x_n) = ((\ell_1 : \ell_2), (\ell_3 : \ell_4))$ for linear forms $\ell_1, \ldots, \ell_4$ in general position.

For $d = 3$ this recovers a component identified earlier by ad hoc methods. For $d \geq 4$ the existence of the component appears to be inaccessible to local deformation arguments: a general member has singularities of arbitrary algebraic multiplicity along two skew lines, and computer-aided calculations show that the component is generically non-reduced.

The next two slides sketch how positive characteristic enters.

Sketch (I): the model

Fix $\mathcal{G}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ with $\mathcal{N}_{\mathcal{G}} = \mathcal{O}(a,b)$ and defining 1-form $\omega \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \Omega^1(a,b))$ whose coefficients are algebraically independent over $\mathbb{Q}$. Set $\mathcal{F} = \pi^*\mathcal{G}$.

Reduced degeneracy on the surface. A result by Mendson produces, for a Zariski dense set of primes $\mathfrak{p}$, a reduction $\mathcal{G}_{\mathfrak{p}}$ that has reduced degeneracy divisor $\Delta_{\mathcal{G}_{\mathfrak{p}}}$.

Pull-back. The map π is unramified over the smooth locus, so

$$\Delta_{\mathcal{F}_{\mathfrak{p}}} = \pi^*\Delta_{\mathcal{G}_{\mathfrak{p}}}$$

is again reduced for the same primes.

Sketch (II): rigidity

The kernel of the p-curvature is exactly the fibres of π. The reduced shape of $\Delta_{\mathcal{F}_\mathfrak{p}}$ and the previous obstruction together force $V(\mathcal{F}_\mathfrak{p})$ to be a p-closed codim-two foliation of degree one, namely the foliation by fibres of $\pi_\mathfrak{p}$. Hence $\delta_2(\mathcal{F}_\mathfrak{p}) = 1$.

Lift to the component. Semi-continuity gives $\delta_2(\Sigma) = 1$ for any component Σ containing $\mathcal{F}$. A general $\mathcal{F}' \in \Sigma$ therefore admits a subfoliation $\mathcal{D}'$ of codim two and degree one. The structure result for codim-one foliations with $\delta_2 = 1$ leaves two options:

1. $\mathcal{F}'$ is defined by a closed rational 1-form without codim-one zeros (excluded by the a, b hypothesis applied to a small deformation), or
2. $\mathcal{F}'$ contains a *unique* algebraically integrable codim-two subfoliation $A_{\mathcal{F}'}$ of degree one.

Identification. For $n = 3$ the foliation $A_{\mathcal{F}'}$ is defined by a single global vector field of degree one. The Jordan structure varies holomorphically in the family; algebraic integrability forces the eigenvalues to be rational, hence locally constant. The model $A_{\mathcal{F}}$ is semi-simple with eigenvalues $(1, 1, -1, -1)$, and this persists in the family.

Other results related to the conjecture

Tangential foliations on $\mathrm{Mor}(\mathbb{P}^1, X)$

X a uniruled projective manifold, $\mathcal{F}$ a codim-one foliation, $M \subset \mathrm{Mor}(\mathbb{P}^1, X)$ an irreducible component containing a free morphism.

Definition
The tangential foliation $\mathcal{F}_{\mathrm{tang}}$ on M is the foliation whose leaves are maximal analytic families of morphisms that do not move points outside the leaves of $\mathcal{F}$ they start on.

A leaf L of $\mathcal{F}_{\mathrm{tang}}$ corresponds to a family of $\mathbb{P}^1$'s in X tangent to $\mathcal{F}$, parameterised by L. The construction is birational over the moduli of rational curves and does not see X outside the uniruled locus.

Detection of the transverse Lie algebra

Theorem (Loray–P.–Touzet, 2020)
Let $\mathcal{F}$ be a codim-one foliation on a uniruled projective manifold X, and L a general leaf of $\mathcal{F}_{\mathrm{tang}}$ on M. If $\delta = \dim \overline{L} - \dim L$ then $\delta \leq 3$. Moreover,

- when $\delta = 3$, $\mathcal{F}$ is transversely projective but not transversely affine.
- when $\delta = 2$, $\mathcal{F}$ is transversely affine but not virtually transversely Euclidean.
- when $\delta = 1$, $\mathcal{F}$ is virtually transversely Euclidean.

The transverse Lie algebra of $\mathcal{F}$ is read off the codimension of a general leaf of $\mathcal{F}_{\mathrm{tang}}$ in its Zariski closure.

One consequence

Corollary. On a simply connected uniruled X, if a general $\mathbb{P}^1$ in M meets every $\mathcal{F}$-invariant algebraic hypersurface transversely and non-trivially, and the general leaf of $\mathcal{F}_{\mathrm{tang}}$ is non-algebraic, then $\mathcal{F}$ is defined by a closed rational 1-form.

In particular, on $\mathbb{P}^n$ many foliations are detected to be transversely projective by the birational geometry of $\mathrm{Mor}(\mathbb{P}^1, \mathbb{P}^n)$.

This is what allowed a partial classification of irreducible components of $\mathrm{Fol}(\mathbb{P}^n, d = 3)$.

Pencils and unlikely intersections

Cerveau's theorem on pencils

Theorem (Cerveau)
Let ω_0, ω_1 be rational 1-forms on $\mathbb{P}^3$ such that $s\omega_0 + t\omega_1$ is integrable for every $(s : t) \in \mathbb{P}^1$. Then either

1. the pencil is a rational pull-back of a pencil on $\mathbb{P}^2$, or
2. every member of the pencil is transversely affine.

This is the Cerveau–Lins Neto dichotomy in the special case of one-parameter integrable families. Cerveau's proof used Lüroth's theorem.

Generalisation (2025)

Theorem (Barbosa–P.)
Let X be a projective manifold and ω_0, ω_1 rational 1-forms on X with $\omega_0 \wedge \omega_1 \neq 0$. If $s\omega_0 + t\omega_1$ is integrable for every $(s:t) \in \mathbb{P}^1$, defining $\mathcal{F}_{(s:t)}$, then either

1. there exist a projective surface Y, a rational map $\pi : X \dashrightarrow Y$ and foliations $\mathcal{G}_{(s:t)}$ on Y with $\mathcal{F}_{(s:t)} = \pi^* \mathcal{G}_{(s:t)}$ for all $(s:t)$, or
2. every $\mathcal{F}_{(s:t)}$ is (singularly) transversely affine.

Beyond pencils

The same work treats $(q+1)$-tuples of foliations whose q-wise intersections coincide. The conclusion is controlled by the transcendence degree

$$\delta \; = \; \mathrm{tr.deg}_{\mathbb{C}} \mathbb{C}(X/\mathcal{G})$$

of the field of rational first integrals of the common codim-q foliation $\mathcal{G}$:

- $\delta = q$: pull-back from a q-dimensional manifold;
- $2 \leq \delta \leq q-1$: at least $q+1-\delta$ of the foliations are transversely projective;
- $\delta \in \{0,1\}$: every foliation in the family is transversely projective.

References

- D. Cerveau, A. Lins Neto, *Irreducible components of the space of holomorphic foliations of degree two in* $\mathbb{CP}(n)$, $n \geq 3$, Ann. of Math. (2) 143 (1996), 577–612.
- D. Cerveau, A. Lins Neto, F. Loray, J.V. P., F. Touzet, *Complex codimension one singular foliations and Godbillon-Vey sequences*, Mosc. Math. J. 7 (2007), 21–54.
- F. Loray, J.V. P., F. Touzet, *Singular foliations with trivial canonical class*, Invent. Math. 213 (2018), 1327–1380.
- F. Loray, J.V. P., F. Touzet, *Deformation of rational curves along foliations*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 21 (2020), 1315–1331.
- W. Mendson, J.V. P., *Codimension one foliations in positive characteristic*, J. Inst. Math. Jussieu 23 (2024), 1705–1750.
- G. S. Barbosa, J.V. P., *Unlikely intersections of codimension one foliations*, preprint, `arXiv:2505.14873` (2025).