Resolution of Foliated varieties by torus actions

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Resolution of singularites and foliations by weighted centers:

[McQ19] M. McQuillan, *Very fast, very functorial, and very easy resolution of singularities*, 2019
[ATW 19] -D. Abramovich, M. Temkin. J. Włodarczyk *Functorial embedded resolution via weighted blowings up* , 2019
[W22] J. Włodarczyk *Functorial resolution by torus actions*, March 2022, arXiv:2007.13846.
[ABTW 25] -D. Abramovich, A.Belotto, M. Temkin. J. Włodarczyk *Principalization on logarithmically foliated orbifolds*

Weights vs smooth centers

- Weights are naturally associated with singularities and their resolution

$$Ex : x^2 + y^3 + z^4, \quad w(x) = \frac{1}{2} \quad w(y) = \frac{1}{3} \quad w(z) = \frac{1}{4}$$

We rescale (multiply by 12) the weights to obtain integers.
$w(x) = 6 \quad w(y) = 4 \quad w(z) = 3$
Weights eliminate many difficulties associated with smooth centers:

- Smooth centers are not compatible with weights.
- They do not usually improve singularities,
- Extra logarithmic structure is needed when resolving by smooth centers

We describe the center as Q-**ideal** (of weight 1)

$$\left(x^{1/6}, y^{1/4}, z^{1/3}\right)$$

Definition: $\mathbb{Q}$-ideals and Rees algebras

$\mathbb{Q}$-**ideals** are the equivalence classes of formal expressions $\mathcal{I}^{\frac{1}{n}}$, where $\mathcal{I}$ is an ideal on X and $n \in \mathbb{N}$.

$$\mathcal{I}^{\frac{1}{n}} \sim \mathcal{J}^{\frac{1}{m}} \quad \text{if} \quad (\mathcal{J}^n)^{\text{int}} = (\mathcal{I}^m)^{\text{int}},$$

where int stands for the integral closure in $\mathcal{O}_X[t]$.
With any $\mathbb{Q}$-ideal $\mathcal{J} = \mathcal{I}^{1/n}$ on X, one can associate **the ideal of sections**:

$$\mathcal{J}_X := \{f \in \mathcal{O}_X \mid f^n \in \mathcal{I}^{\text{int}}\}.$$

It is exactly the t-gradation of the algebra $\mathcal{A}_{\mathcal{J}} = \mathcal{O}_X[\mathcal{I}t^n]^{\text{int}}$.

This determines the **graded Rees algebra of ideals**

$$\mathcal{A}_{\mathcal{J}} = \mathcal{O}_X[\mathcal{I}t^n]^{\mathrm{int}} = (\mathcal{O}_X[\mathcal{J}t])_X := \bigoplus (\mathcal{J}^i)_X t^i$$

on X.

Any $\mathbb{Q}$-ideal of the form $\mathcal{J} = \mathcal{I}^{1/n}$ on a normal scheme X determines a unique blow-up

$$\pi : Y = \mathcal{P}roj(\mathcal{A}_{\mathcal{J}}) \to X$$

of $\mathcal{J}$ on X, which can be understood as the normalized blow-up of $\mathcal{I}$ on X.
It transforms $\mathcal{J} = \mathcal{I}^{1/n}$ into a $\mathbb{Q}$-Cartier divisor

$$(\mathcal{O}_X \cdot \mathcal{I})^{1/n} = (\mathcal{O}_X(-E))^{1/n}$$

for a Cartier exceptional divisor E.

Rational Rees Algebras

Definition: A *rational Rees algebra* is a finitely generated $\mathcal{O}_X$-algebra:

$$R = \bigoplus_{a \in \Gamma} R_a t^a \subset \mathcal{O}_X[t^{1/w_R}],$$

where:

- $\Gamma \subset \mathbb{Q}_{\geq 0}$ is a finitely generated additive subsemigroup.
- $R_0 = \mathcal{O}_X$ and $R_a \cdot R_b \subseteq R_{a+b}$.
- $w_R \in \mathbb{Q}_{>0}$ is the *grading denominator* (minimal value such that $\Gamma \subseteq (1/w_R) \cdot \mathbb{Z}_{\geq 0}$).

Key Concepts:

- **Extended Rees algebra**: $R^{\mathrm{ext}} := R[t^{-1/w}]$ (for any multiple w of w_R).
- **Integral closures**: R^{int} in $\mathcal{O}_X[t^{1/w_R}]$ and R^{Int} in $\mathcal{O}_X[t^{1/w}]$.
- **Vertex**: $V(R) = V(R^{\mathrm{ext}}) := V\left(\sum_{a>0} R_a\right)$.

Examples & Rees Centers

Standard Rees Algebra: For an ideal $\mathcal{I}$, the standard $\mathbb{Z}$-graded algebra is:

$$\mathcal{A}_{\mathcal{I}} := \mathcal{O}_X[\mathcal{I}t] = \bigoplus_{n\geq 0} \mathcal{I}^n t^n, \quad \mathcal{A}_{\mathcal{I}}^{\text{ext}} = \mathcal{O}_X[t^{-1}, \mathcal{I}t].$$

Rees Centers: A *Rees center* is locally of the form:

$$\mathcal{A} = \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\text{int}},$$

where $x_1, \ldots, x_k$ are part of a local parameter system, $a_i \in \mathbb{Q}_{>0}$, and the integral closure is in $\mathcal{O}_X[t^{1/w_A}]$ with $w_A := \operatorname{lcm}(a_1, \ldots, a_k)$.
Extended Center:

$$\mathcal{A}^{\text{ext}} := \mathcal{O}_X[t^{-1/w}, x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}],$$

with w a multiple of w_A ensuring all w/a_i are integral.

Rescaling & Monomial Valuations

Rescaling: For $w_0 \in \mathbb{Q}_{>0}$, the rescaling of R is:

$$R^{w_0} := \bigoplus_{a \in \Gamma} R_a t^{w_0 a} \subset \mathcal{O}_X[t^{w_0/w_R}].$$

Lemma

The map $ft^a \mapsto ft^{w_0 a}$ defines an isomorphism $R \simeq R^{w_0}$. Thus, R is integrally closed iff R^{w_0} is.

Monomial Valuations: If $V(\mathcal{A})$ is irreducible, there exists a unique monomial valuation ν_A with $\nu_A(x_i) = 1/a_i$ such that:

$$\mathcal{A}_a = \{f \in \mathcal{O}_X \mid \nu_A(f) \geq a\}.$$

Lemma For $\mathcal{A}^{\text{ext}} = \mathcal{O}_X[t^{-1/w}, x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]$, the integral closure of $\mathcal{A} \subset \mathcal{O}_X[t^{1/w}]$ is $\mathcal{A}^{\text{Int}} = (\mathcal{A}^{\text{ext}})_{\geq 0}$.

Regular Centers vs. Rees Centers

A regular $\mathbb{Q}$-ideal center $\mathcal{J} = (u_1^{1/w_1}, \ldots, u_k^{1/w_k})$ with $w_i \in \mathbb{N}$ corresponds to:

- the Rees integral algebra $\mathcal{A}_{\mathcal{J}} = \mathcal{O}_X[u_1 t^{w_1}, \ldots, u_k t^{w_k}]^{\text{int}}$,
- and the extended Rees integral algebra $\mathcal{A}_{\mathcal{J}}^{\text{ext}} = \mathcal{O}_X[t^{-1/w}, u_1 t^{w_1}, \ldots, u_k t^{w_k}]$.

More generally with $\mathbb{Q}$-ideal center

$$\mathcal{J} = (u_1^{a_1}, \ldots, u_k^{a_k}), \quad a_i \in Q$$

we can associate a rational Rees algebra

$$\mathcal{A}_{\mathcal{J}}^{\text{ext}} = \mathcal{O}_X[t^{-1/w}, u_1 t^{1/a_1}, \ldots, u_k t^{1/a_k}].$$

After fixing a chosen multiple w of $w_A = \mathrm{lcm}(a_1, \ldots, a_k)$ there is a natural correspondence between:

- Rees centers $\mathcal{A} = \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{Int}}$, with $a_i \in \mathbb{Q}_{>0}$.
- Extended Rees centers $\mathcal{A}^{\mathrm{ext}} = \mathcal{O}_X[t^{-1/w}, x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]$.
- Their associated rescaled integral Rees algebras $\mathcal{A}_{\mathcal{J}} = \mathcal{A}^w = \mathcal{O}_X[u_1 t^{w_1}, \ldots, u_k t^{w_k}]^{\mathrm{int}}$, where $w_i = w/a_i$.
- Their associated extended integral Rees algebras

$$\mathcal{A}^{\mathrm{ext}}_{\mathcal{J}} = (\mathcal{A}^{\mathrm{ext}})^w = \mathcal{O}_X[t^{-1}, u_1 t^{w_1}, \ldots, u_k t^{w_k}].$$

- Their associated $\mathbb{Q}$-ideal centers $\mathcal{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$ with $w_i \in \mathbb{N}$.

Weighted blow-ups of centers

In particular, the center $\mathcal{J} := (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$ with *integral weights* w_i on a regular scheme X determines the **weighted blow-up**:

$$Y = \mathcal{P}roj(\mathcal{A}_{\mathcal{J}}) = \mathcal{P}roj(\mathcal{O}_X[t^{w_1}x_1, \ldots, t^{w_k}x_k]^{\mathrm{int}}).$$

The gradations $A_{\mathcal{J}i}$ of the Rees algebra

$$A_{\mathcal{J}} = \bigoplus A_{\mathcal{J}i}t^i := \mathcal{O}_X[t^{w_1}x_1, \ldots, t^{w_k}x_k]^{\mathrm{int}}$$

represent sections of $A_{\mathcal{J}i} = (\mathcal{J}^i)_X$ of $\mathcal{J}^i = \left(x_1^{\frac{1}{w_1}} \cdots x_k^{\frac{1}{w_k}}\right)^i$.

The stack-theoretic weighted blow-ups of centers

The stack-theoretic blow-up of any $\mathbb{Q}$-ideal $\mathcal{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$ is defined to be the stack-theoretic quotient of the scheme:

$$[(\mathrm{Spec}_X((\mathcal{O}_X[\mathcal{J}t])_X \setminus V((\mathcal{J}t)_X))/\mathbb{G}_m] =$$

$$[(\mathrm{Spec}_X(\mathcal{O}_X[t^{w_1}x_1, \ldots, t^{w_k}x_k]^{\mathrm{int}}) \setminus V(t^{w_1}x_1, \ldots, t^{w_k}x_k))/\mathbb{G}_m].$$

with the natural action of the multiplicative group $\mathbb{G}_m := \mathrm{Spec}(k[t, t^{-1}])$. Observe that the space

$$\mathrm{Spec}_X(\mathcal{O}_X[t^{w_1}x_1, \ldots, t^{w_k}x_k]^{\mathrm{int}}) = \mathrm{Spec}_X(\mathcal{O}_X[t^{a_1}x_1, \ldots, t^{a_k}x_k \mid c_i \leq w_i])$$

is usually singular with normal toric singularities (even for smooth blow-ups).

Description of weighted blow-ups

$$Y = [(\mathrm{Spec}_X(\mathcal{O}_X[t^{w_1}x_1, \ldots, t^{w_k}x_k]^{\mathrm{int}}) \setminus V(t^{w_1}x_1, \ldots, t^{w_k}x_k))/\mathbb{G}_m] \to X$$

of $\mathcal{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$.

We consider the charts $x_i t^{w_i} \neq 0$ as for the smooth blow-ups. In the chart $x_1 t^{w_1} \neq 0$, we consider the section $x_1 t^{w_1} = 1$ and the group $\mu_1 = \{t^{w_1} = 1\} \subset \mathbb{G}_m$. Then on the section $x_1 = t^{-w_1} = y^{w_1}$ is the power of the exceptional divisor $y = t^{-1}$.

Moreover, we obtain the associated transformation $x_1 = y_1^{k_1}$ with the finite group action:

$$\begin{aligned} x_1' &= y^{w_1} \\ x_2' &= x_2' y^{w_2} \\ &\ldots \\ x_k' &= x_k' y^{w_k} \end{aligned}$$

Singularities and weights

Example: $f = x_1^{a_1} + \ldots + x_k^{a_k}$ on $\mathbb{A}^n$. We associate the to x_i the rational weights $1/a_i$, and rescaled them to integral (positive) weights :

$$w(x_i) := w_i, \quad \text{where} \quad a_1 w_1 = \ldots = a_k w_k$$

Note that

$$(w_1, \ldots, w_k) \sim (1/a_1, \ldots, 1/a_k), \quad w_i = w \cdot 1/a_i, \quad w = lcm(a_1, \ldots, a_k)$$

We describe the center as a $\mathbb{Q}$-**ideal** (of weight 1):

$$\left(x_1^{1/w_1}, \ldots, x_k^{1/w_k}\right)$$

After blow-up of $\mathcal{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$, the singularity $x_1^{a_1} + \ldots + x_k^{a_k}$ is transformed in the chart $x_1 t^{w_1} \neq 0$ into: $y^{a_1 w_1}(1 + (x_2')^{a_2} + \ldots + (x_k')^{a_k})$ with the nonsingular strict transform: $1 + (x_2')^{a_2} + \ldots + (x_k')^{a_k}$.

Cobordant Blow-Up and Exceptional Divisor

Definition:[W22] The *Full Cobordant Blow-Up of* $\mathcal{J} = (x_1^{1/w_1}, \dots, x_k^{1/w_k})$ on a regular X is defined by

$$B = \mathrm{Spec}_X\big(\mathcal{O}_X[t^{-1}, x_1 t^{w_1}, \dots, x_k t^{w_k}]\big) \to X,$$

Vertex of B**:**

$$V = \mathrm{Vert}(B) := V(x_1 t^{w_1}, \dots, x_k t^{w_k}).$$

This is called the **vertex** of B, analogous to the vertex of an affine cone over a projective scheme.

Cobordant Blow-Up:

The T-invariant morphism:

$$\sigma_+ : B_+ = B \setminus \mathrm{Vert}(B) \to X.$$

Trivial Cobordant Blow-Up:

$B_- = B \setminus V(t^{-1}) = \mathrm{Spec}_X(\mathcal{O}_X[t, t^{-1}]) \to X$, where $D := V_B(t^{-1})$ is the exceptional divisor.

The Exceptional Divisor

Pulling back the center $\mathcal{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$ to B_+ factors out t^{-1}:

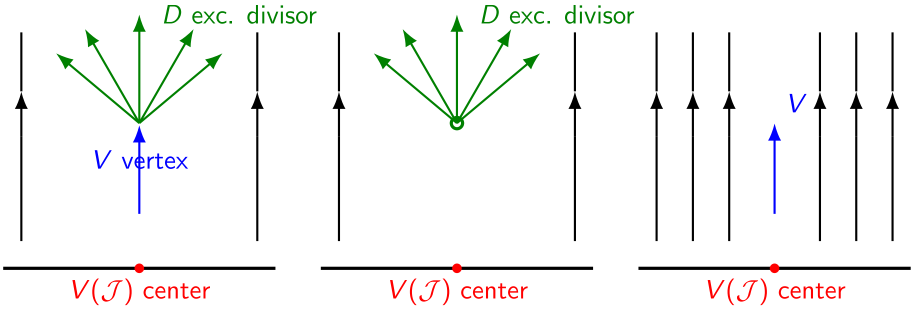
$$\mathcal{O}_{B_+} \cdot \mathcal{J} = \mathcal{O}_{B_+} t^{-1} \cdot \underbrace{(x_1^{1/w_1} t, \ldots, x_k^{1/w_k} t)}_{\text{defines the vertex } V}$$

The Magic of Removing the Vertex:

- By definition, the cobordant space is $B_+ = B \setminus V$.
- Since the vertex V is removed, the remaining ideal has no common zeros anywhere on B_+.
- Therefore, it generates the unit ideal: $\mathcal{O}_{B_+}(\ldots) = \mathcal{O}_{B_+}$.

Conclusion: The pullback is strictly principal.

$$\mathcal{O}_{B_+} \cdot \mathcal{J} = \mathcal{O}_{B_+} t^{-1}$$



B full cobord. blow-up $B_+ = B \setminus V$ cob. blow-up $B \setminus D = X \times \mathbb{G}_m$

Figure: Cobordant blow-up: the role of the vertex V and the exceptional divisor D.

Full Cobordant blow-up: local equations

The **full cobordant blow-up** $B = \mathrm{Spec}_X(\mathcal{O}_X[t^{-1}, x_1 t^{w_1}, \ldots, x_k t^{w_k}]) \to X$ of $\mathcal{J}$, can be represented locally as a closed subspace $X \times \mathrm{A}^{k+1}$ with coordinates $x_1, \ldots, x_n, x_1' = x_1 t^{w_1}, , \ldots, x_k' = x_1 t^{w_1}, s = t^{-1}$

$$B = \mathrm{Spec}(\mathcal{O}_X[s, x_1', \ldots, x_n']/((x_1' s^{w_1} - x_1), \ldots, (x_k' s^{w_1} - x_k)].$$

Consequently, the full cobordant blow-up $B \to X$ can be described by a single chart with the following coordinates:

- $s = t^{-1}$ is the inverse of the coordinate t representing the action of torus $G_m = \mathrm{Spec}(K[t, t^{-1}])$.
- $x_i' = x_i \cdot t^{w_i} = x_i / s^{w_i}$ for $1 \leq i \leq k$, and
- $x_j' = x_j$ for $j > k$.

Weighted blow-ups via Cobordant blow-ups

The cobordant blow-ups is not a birational transformation. It introduces the action of torus T.
One can recover the standard definition of the *weighted blow-up* to be

$$B_+/T \to X,$$

where B_+/T is a *geometric quotient* (space of orbits) and *stack-theoretic weighted blow-up*

$$[B_+/T] \to X,$$

for the stack-theoretic quotient $[B_+/T]$.

Remark. Weighted stack-theoretic blow-ups were introduced in the context of resolution of foliations by Panazzolo and in the resolution of varieties independently by McQuillan-Marzo **[McQ19]** and by Abramovich-Temkin-Włodarczyk **[ATW19]**.

Cobordant vs. Orbifold Weighted Blow-ups

Cobordant blow-ups were introduced in [W22] and [W23], and independently formulated as presentations of stacky weighted blow-ups by Quek and Rydh in [QR22].

Key Advantages Over Stacky Blow-ups:

- **Elimination of Stacks:** It replaces the abstract 2-categorical machinery of stacks with classical scheme theory equipped with a simple torus action.
- **Coordinate Simplicity:** It provides a global presentation requiring *only one chart*, bypassing the tedious gluing of multiple orbifold patches.
- **Foliation Compatibility:** It guarantees much better geometric properties when transforming and resolving derivations on foliated varieties.

Historical and Geometric Connections

- **Algebraic Origin:** The definition originates from the concept of the **extended Rees algebra**, introduced by Rees.
- **Geometric Origin:** It naturally relates to the classical construction of the **deformation to the normal cone**, as developed and studied by MacPherson, Fulton, Reid, Kollár, and others.
- **Birational Cobordism:** The approach is fundamentally tied to the theory of **birational cobordism** (W), utilizing smooth spaces equipped with torus actions to represent and resolve complex birational morphisms.

Example: Cobordant Blow-up

Example: Consider the singularity $f = x_1^{a_1} + \ldots + x_k^{a_k}$ on $\mathbb{A}^n$.

Consider the center defined by the rational Rees algebra:

$$\mathcal{A} = \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\text{int}}$$

By rescaling the weights to integers w_i such that $a_1 w_1 = \cdots = a_k w_k = w$, we obtain the full cobordant blow-up:

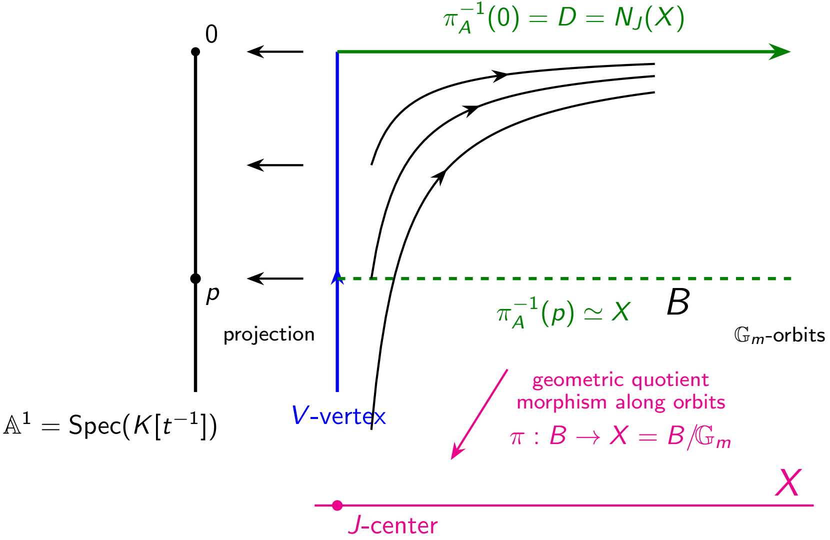
$$B = \mathrm{Spec}_X \left(\mathcal{O}_X[t^{-1}, x_1 t^{w_1}, \ldots, x_k t^{w_k}] \right)$$

Using the canonical coordinates $x_i' = x_i t^{w_i}$ on B, the equation lifts as:

$$\begin{aligned} f = x_1^{a_1} + \ldots + x_k^{a_k} &= t^{-w} \left((x_1 t^{w_1})^{a_1} + \ldots + (x_k t^{w_k})^{a_k} \right) \\ &= \underbrace{t^{-w}}_{\text{exceptional}} \cdot \underbrace{\left(\left(x_1'\right)^{a_1} + \ldots + \left(x_k'\right)^{a_k} \right)}_{\sigma^s(f) \text{ -- strict transform}} \end{aligned}$$

Notice that the strict transform $\sigma^s(f)$ has the exact same equation as f. Thus, removing the vertex $V(x_1', \ldots, x_k')$ yields a smooth space!

Cobordant Blow-Up and Deformation to the Normal Bundle



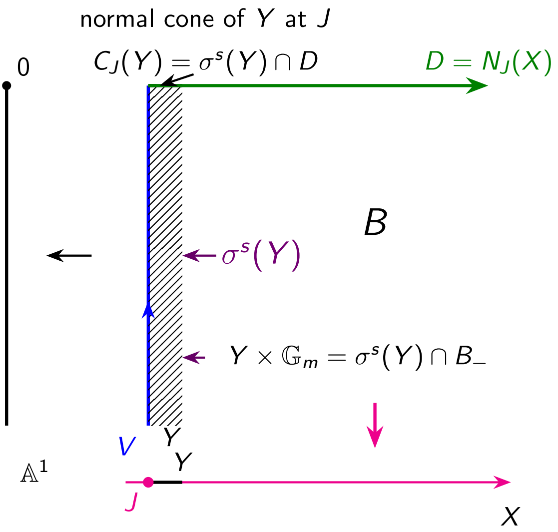


Figure: The strict transform $\sigma^s(Y)$ of $Y \subset X$ and its decomposition

Single blow-up resolution

Theorem

[W22] *(Any characteristic) Let J be the weighted center on X such that $V(J) \subseteq Y \subset X$.*

Assume that:

- $\mathrm{Sing}(Y) \subset V(J)$
- $\mathrm{Sing}\,(C_J(Y) \subseteq V(\mathrm{in}(J)) \subset N_J(X))$

Then the strict transform $\sigma^s(Y)$ of Y under the weighted cobordant blow-up $B_+ \to X$ is regular.

Generalized resolution principle **[W22]** (Any characteristic)

Let $Y \subset X$ be as above. Let $Inv : Y \to \Gamma$ be an invariant with the values in an ordered set Γ. **Monotonicity condition** :

- for any smooth $X' \subset X$, and $Y' = Y \cap X$, and $p \in Y$: $Inv_p(Y') \geq Inv_p(Y)$.

Denote by $MaxInv(Y)$ the support of all the points on Y where Inv attains its maximum.

Assume that J be a weighted center such that $V(\mathcal{J}) \subseteq Y \subset X$

- $MaxInv(Y) \subset V(J)$
- $Maxinv_p\,(C_J(\mathcal{I}_Y) \subseteq V(\mathrm{in}(J)) \subset N_J(X))$

Then for the strict transform $\sigma^s(Y)$ of Y under the weighted cobordant blow-up $B_+ \to X$ the maximum value of the invariant Inv drops.

Example-Brieskorn singularities in any characteristic

Let $Y \subset X = \mathbb{A}^n_K$ be described as:

$$a_1 x_1^{b_1} + \ldots + a_n x_n^{b_n}, \quad a_i \in K$$

$$\mathrm{inv}_0(Y) = (b_1, \ldots, b_n)$$

with the center

$$\mathcal{J} = \mathcal{O}_X[x_1 t^{1/b_1}, \ldots, x_k t^{1/b_n}]^{\mathrm{int}}.$$

Rescaling gives $B = \mathrm{Spec}_X(\mathcal{O}_X[t^{-1}, x_1 t^{w_1}, \ldots, x_k t^{w_n}])$

$$\begin{aligned} \mathcal{O}_B \cdot \mathcal{I}_Y &= t^{-b_1 w_1} \left(a_1 \left(x_1 t^{w_1}\right)^{b_1} + \ldots + a_n \left(x_n t^{w_n}\right)^{b_n}\right) \\ &= t^{-b_1 w_1} \underbrace{\left(a_1 \left(x_1'\right)^{b_1} + \ldots + a_n \left(x_n'\right)^{b_n}\right)}_{\sigma^s(\mathcal{I}_Y)\text{ - controlled and strict transform}} . \end{aligned}$$

Example: Brieskorn Singularities (1/2)

Let $Y \subset X = \mathbb{A}^n_K$ be defined by the Brieskorn equation:

$$f = a_1 x_1^{b_1} + \ldots + a_n x_n^{b_n}$$

Assume $\mathrm{Sing}(Y) \subset V(x_1, \ldots, x_n)$. *(This holds automatically in characteristic 0).*

1. The Rational Center

We define the maximal admissible center using the rational Rees algebra:

$$\mathcal{A} = \mathcal{O}_X[x_1 t^{1/b_1}, \ldots, x_n t^{1/b_n}]^{\mathrm{int}}$$

2. The Cobordant Space

We rescale the weights to integers w_i such that $b_1 w_1 = \cdots = b_n w_n = w$. This gives us the full cobordant space B:

$$B = \mathrm{Spec}_X \left(\mathcal{O}_X[t^{-1}, x_1 t^{w_1}, \ldots, x_n t^{w_n}] \right)$$

Example: Brieskorn Singularities (2/2)

3. Lifting the Ideal

We lift the ideal $\mathcal{I}_Y$ to B using the canonical invariant coordinates $x_i' = x_i t^{w_i}$. Factoring out the exceptional divisor yields:

$$\mathcal{O}_B \cdot \mathcal{I}_Y = \underbrace{t^{-w}}_{\text{exc. div}} \cdot \underbrace{\left(a_1(x_1')^{b_1} + \ldots + a_n(x_n')^{b_n}\right)}_{\sigma^s(\mathcal{I}_Y)\text{ – strict transform } Y'}$$

4. The Geometric Resolution

Notice that the strict transform Y' has the **exact same equation** as Y.

Because the original singularities were confined to the origin:

$$\mathrm{Sing}(Y) \subset V(x_1, \ldots, x_n)$$

It follows directly that the singularities of the strict transform are confined to the vertex of B:

$$\mathrm{Sing}(Y') \subset V(x_1', \ldots, x_n') = V$$

Homogeneous Subschemes

Let $X = \mathbb{A}^n_V = \mathrm{Spec}_V(\mathcal{O}_V[x_1, \ldots, x_n])$.

Induced Grading: A weighted center $\mathcal{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$ naturally grades the ring $\mathcal{O}_X$ via the map:

$$\phi : \mathcal{O}_V[x_1, \ldots, x_n] \longrightarrow \mathcal{O}_V[t^{w_1}x_1, \ldots, t^{w_k}x_k, x_{k+1}, \ldots, x_n]$$

where $x_i \mapsto t^{w_i}x_i$ for $i \leq k$, and $x_j \mapsto x_j$ for $j > k$. A monomial of degree $d = \sum a_i w_i$ maps to $t^d x^\alpha$.

Strict Transform Connection: If $f \in \mathcal{J}^d \setminus \mathcal{J}^{d+1}$ is homogeneous of degree d, then $\phi(f) = t^d f$. This perfectly matches the **strict transform** of f in the cobordant algebra $\mathcal{O}_B$.

Definition

A closed subscheme of X defined by a homogeneous ideal $\mathcal{I} \subset \mathcal{O}_X$ (with respect to the $\mathcal{J}$-grading) is called a **homogeneous subscheme** with respect to $\mathcal{J}$.

Structural Lemma for Homogeneous Ideals

Because $\mathcal{I}$ is homogeneous, the cobordant space possesses a remarkable trivialization.

Lemma

Let $\mathcal{I} \subset \mathcal{O}_X$ be homogeneous w.r.t. $\mathcal{J}$, and let $\sigma : B \to X$ be the cobordant blow-up. There is a canonical isomorphism:

$$B \simeq X \times \mathbb{A}^1 = \mathrm{Spec}_V(\mathcal{O}_V[x_1, \ldots, x_n, y])$$

defined by the coordinate mapping:

$$x_i \mapsto t^{w_i} x_i \quad (i \leq k), \qquad x_j \mapsto x_j \quad (j > k), \qquad y \mapsto t^{-1}.$$

Geometric Consequences under this isomorphism:

- The pullback ideal $\mathcal{O}_{X \times \mathbb{A}^1} \cdot \mathcal{I}$ maps exactly to the strict transform $\sigma^s_B(\mathcal{I})$ on B.
- The locus $V(x_1, \ldots, x_n) \subset X \times \mathbb{A}^1$ maps precisely to the **vertex** V of

Order and rational Rees algebras (W)

Definition: The *order* of an ideal $\mathcal{I}$ at a point $p \in X$ is given by:

$$\operatorname{ord}_p(\mathcal{I}) = \max\{a \in \mathbb{Z}_{>0} \mid \mathcal{I} \subset m_p^a\},$$

where $m_p \subset \mathcal{O}_{X,p}$ is the maximal ideal of p in the local ring $\mathcal{O}_{X,p}$.

Introducing Rees algebra with dummy variable t write:

$$\operatorname{ord}_p(\mathcal{I}) = \max\{a \in \mathbb{Q}_{>0} \mid \mathcal{I}t^a \subset \mathcal{O}_X[m_p t]\}.$$

By rescaling $t \mapsto t^{1/a}$ using rational Rees algebras:

$$\operatorname{ord}_p(\mathcal{I}) = \max\{a \in \mathbb{Q}_{>0} \mid \mathcal{I}t \subset \mathcal{O}_X[m_p t^{1/a}]\}.$$

Order of Rees Algebras

Definition (Order of an Element)

Let $R = \bigoplus R_a t^a$ be a Rees algebra. For any homogeneous element $f \in R_b$, the **order** at a point $p \in X$ is defined as:

$$\operatorname{ord}_p(ft^b) := \frac{\operatorname{ord}_p(f)}{b}$$

Definition (Order of the Algebra)

The **order of the Rees algebra** R at the point p is the infimum of the orders of its positive-degree components:

$$\operatorname{ord}_p(R) := \min_{a \in \Gamma_{>0}} \left\{ \frac{\operatorname{ord}_p(R_a)}{a} \right\}$$

where $\Gamma_{>0}$ denotes the set of degrees $a > 0$ such that $R_a \neq 0$.

Lemma

The classical order of an ideal $\mathcal{I}$ at a point p perfectly coincides with the order of its associated standard Rees algebra:

$$\mathrm{ord}_p(\mathcal{I}) = \mathrm{ord}_p\left(\mathcal{O}_X[\mathcal{I}t]\right) = \max\{a \in \mathbb{Q}_{>0} \mid \mathcal{I}t \subset \mathcal{O}_X[m_p t^{1/a}]\}.$$

$$\mathrm{ord}_p(R) = \max\{a \in \mathbb{Q}_{>0} \mid R \subset \mathcal{O}_X[m_p t^{1/a}]\}.$$

Semicontinuity of the Order

Lemma

The order functions $p \mapsto \mathrm{ord}_p(\mathcal{I})$ and $p \mapsto \mathrm{ord}_p(R)$ are upper semicontinuous. Thus, for any constant a, the locus $\mathrm{supp}(\mathrm{ord} \geq a)$ is strictly Zariski closed.

- **For Ideals:** The geometric condition $\mathrm{ord}_p(\mathcal{I}) \geq a$ is identically equivalent to the vanishing of all lower-order derivatives:

$$(D^\alpha f)(p) = 0 \quad \text{for all } f \in \mathcal{I}, \ |\alpha| < a$$

  This is precisely the zero locus of an ideal of derivatives, which is closed.

- **For Rees Algebras:** If $R = \bigoplus R_k t^k$, then $\mathrm{ord}_p(R) \geq a$ requires:

$$\mathrm{ord}_p(R_k) \geq ka \quad \text{for all } k > 0$$

  This is simply an intersection of closed sets, which remains closed.

Definition of the Invariant via Rees Algebras

Generalizing the Order of an Ideal:

$$\mathrm{inv}_p(\mathcal{I}) := \max\left\{(a_1, \ldots, a_k) \mid \mathcal{I}t \subset \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{int}}\right\}$$

where the rational weights are ordered lexicographically ($a_1 \leq \ldots \leq a_k$).

If $\mathrm{inv}_p(\mathcal{I}) = (a_1, \ldots, a_k)$ and the algebra $\mathcal{A} = \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{int}}$ satisfies $\mathcal{A} \supset \mathcal{I}t$, then $\mathcal{A}$ is called a *maximal admissible center* for $\mathcal{I}$.

Extension to Rational Rees Algebras: For a rational Rees algebra $R = \bigoplus_{a \in \Gamma} R_a t^a$, where $\Gamma \subset \mathbb{Q}_{\geq 0}$ is a finitely generated additive subgroup:

$$\mathrm{inv}_p(R) := \max\left\{(a_1, \ldots, a_k) \mid R \subset \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{int}}\right\}$$

Definition If $\mathrm{inv}_p(R) = (a_1, \ldots, a_k)$ and $\mathcal{A} = \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{int}} \supset R$, then $\mathcal{A}$ is called a *maximal admissible center* for R.

The Invariant and the Order

Suppose the maximal invariant of an ideal $\mathcal{I}$ at a point p is given by:

$$\mathrm{inv}_p(R) = (a_1, \ldots, a_k) = \max\left\{(a_1, \ldots, a_k) \mid R \subset \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{int}}\right\}$$

where the weights are ordered such that $a_1 \leq \ldots \leq a_k$.

The Classical Order: Recall :

$$\mathrm{ord}_p(R) = \max\left\{a \mid R \subset \mathcal{O}_X[x_1 t^{1/a}, \ldots, x_n t^{1/a}]^{\mathrm{int}}\right\}$$

The Connection: Because a_1 is the smallest weight in the optimal $\mathcal{F}$-aligned presentation, it acts as the universal bound for the uniform classical order. Thus we have:

$$\mathrm{ord}_p(R) = a_1$$

Extended Rees Algebra and Cobordant Blow-Up

Consider an ideal $\mathcal{I}$ or a Rees algebra R satisfying the admissibility relation:

$$\mathcal{I}t \quad (\text{or } R) \subset \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\text{int}}$$

Extended Rees Algebra: We adjoin the inverse parameter to form the extended algebra:

$$\mathcal{A}^{\text{ext}} = \mathcal{O}_X[t^{-1/w_A}, x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]$$

where $w_A := \operatorname{lcm}(a_1, \ldots, a_k)$ is the least common multiple of the rational orders.

Rescaled Algebra: By rescaling the grading to integers using $w_i = w_A/a_i$, we obtain the algebra that defines the **cobordant space** B:

$$\mathcal{O}_B = \mathcal{O}_X[t^{-1}, x_1 t^{w_1}, \ldots, x_k t^{w_k}]$$

where $B = \text{Spec}_X(\mathcal{O}_B)$.

The Hironaka maximal contact and the order of ideal

Consider the maximal admissibility relation

$$\mathcal{I}t^a \subset \mathcal{O}_X[m_p t]$$

, where $a = \mathrm{ord}_p(\mathcal{I})$. Acting by $t^{-1}\mathcal{D}_X = \mathrm{span}(t^{-1}\partial_{x_i})$ we enlarge left side and preserve the right side

$$\mathcal{O}_X[m_p t] = \mathcal{O}_X \oplus m_p t \oplus m_p^2 t^2 \ldots .$$

In gradation 1 we obtain

$$\mathcal{D}^{a-1}(\mathcal{I})t \subset m_p t = (x_1, \ldots, x_n)t$$

. Thus there is a coordinate $x_1 \in \mathcal{D}^{a-1}(\mathcal{I})$ called *Hironaka maximal contact*, which is up to a coordinate change one of the coordinate generators of the right side:

$$m_p t = (x_1, \ldots, x_n)t$$

Properties of the Hironaka Maximal Contact

Let $R = \bigoplus R_a t^a$ be a Rees algebra of order a_1 at a point p.

The **maximal contact element** $x_1 t^{1/a_1}$ is systematically constructed by acting with the rescaled derivations $t^{-1/a_1}\mathcal{D}_X$ on the graded components $R_a t^a$.

This element satisfies two fundamental properties:

1. **Algebraic Coordinate:** It serves as the *first coordinate* in the construction of the maximal admissible center:
$$R \subset \mathcal{A}_{\mathcal{J}} = \mathcal{O}_{X,p}[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\text{int}}$$
2. **Geometric Bounding:** The locus of points where the order of the algebra remains at least a_1 is strictly contained within the smooth hypersurface defined by the maximal contact:
$$\operatorname{supp}\big(\operatorname{ord}(R) \geq a_1\big) \subset V(x_1)$$

Coefficient ideals

Given any Rees algebra $R = \mathcal{O}_X[f_j t^{b_j}]$, the **coefficient ideal** $C_{x_1 t^{1/a_1}}(R)$ of R with respect to $x_1 t^{1/a_1}$ is a Rees $\mathcal{O}_{V(x_1)}$- algebra generated by the graded coefficients $c_{fi}(y) t^{b_j - i/a_1}$ of the generators

$$f_j t^{b_j} = \sum c_{ji}(y) x_1^i t^{b_j} \in R_{b_j} t^{b_j} = \sum c_{ji}(y) t^{b_j - i/a_1} (x_1 t)^i$$

$$C_{x_1 t^{1/a_1}}(R) = \mathcal{O}_{V(\overline{x}_1)}[c_{ji}(y) t^{b_j - i/a_1} \mid \quad i/a_1 < b_j]$$

for the coordinate system $(x_1, y) = (x_1, \ldots, x_n)$.

Reduction Property of the Coefficient Ideal

- Let $R = \bigoplus R_a t^a = \mathcal{O}_X[f_j t^{b_j}]$ be the Rees algebra of order a_1 at a point $p \in X$, with maximal contact given by $x_1 t^{1/a_1}$.
- Let $\mathcal{A}$ be the maximal center for R at p.

Then:

- The restriction $\mathcal{A}_{|V(x_1)}$ is a maximal admissible center for the coefficient ideal $C_{x_1 t^{1/a_1}}(R)$.
- $\mathcal{A} = \mathcal{O}_X \left[x_1 t^{1/a_1}, \mathcal{A}_{|V(x_1)}\right]^{\mathrm{int}}$

Proof This follows from the fact that the generators of R:

$$f_j t^{b_j} = \sum c_{ji}(y) t^{b_j - i/a_1} (x_1 t)^i \in \mathcal{A}$$

if and only if the corresponding generators of the coefficient ideal satisfy:

$$c_{ji}(y) t^{b_j - i/a_1} \in \mathcal{A}_{|V(x_1)} \subset \widehat{\mathcal{O}}_{X,p} \cdot \mathcal{A}$$

Construction of the Maximal Admissible Center

The construction of the maximal admissible center of $\mathcal{I}$ at a point p is performed recursively. Define:

$$R_1 := \mathcal{O}_X[\mathcal{I}t] \subset \mathcal{A}_1 := \mathcal{A}_{\mathcal{J}}$$

and for $i = 1, \ldots, k$:

$$R_{i+1} := C_{x_i t^{1/a_i}}(R_i) \subset \mathcal{A}_{i+1} := \mathcal{A}_{i|V(x_i)}$$

$$R \subset \mathcal{A}_{\mathcal{J}} = \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_i t^{1/a_i}, \mathcal{A}_{i+1}]^{\mathrm{int}} \textit{max. admissible}$$

The process ends with:

$$R_{k+1} = 0, \quad \mathcal{A}_{k+1} = 0, \quad R \subset \mathcal{A}_{\mathcal{J}} = \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{int}}$$

maximal admissible

At each step:

- $a_i := \operatorname{ord}_p(R_i)$
- $x_i t^{1/a_i}$ is a maximal contact element for R_i

Recursive Structure and Invariant

Recursive structure of the Rees algebra:

$$\mathcal{A}_{\mathcal{J}} = \mathcal{A}_1 = \mathcal{O}_X[x_1 t^{1/a_1}, \mathcal{A}_2]^{\mathrm{int}} = \cdots = \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{int}}$$

This yields a unique maximal ideal defined by the sequence of orders and maximal contact elements for the Rees algebras R_i.

Alternative definition of the invariant at p:

$$\mathrm{inv}_p(\mathcal{I}) = (\mathrm{ord}_p(R_1), \ldots, \mathrm{ord}_p(R_k)) = (a_1, \ldots, a_k),$$

where for each i:

$$\begin{cases} R_1 = \mathcal{O}_X[\mathcal{I}t] \\ \mathrm{ord}_p(R_i) = a_i \quad \text{(order)} \\ x_i \in \mathcal{D}^{a_i b - 1}(f), \quad ft^b \in R_i, \quad \mathrm{ord}_p(ft^b) = \frac{\mathrm{ord}_p(f)}{b} = a_i \quad \text{(max. contact)} \\ R_{i+1} = C_{x_i t^{1/a_i}}(R_i) \quad \text{(coefficient ideal)} \end{cases}$$

The Inductive Principle for the Invariant

The construction of the maximal contact allows us to compute the invariant tuple inductively:

$$\mathrm{inv}_p(R) = \Big(\mathrm{ord}_p(R),\, \mathrm{inv}_p\big(C_{x_1 t^{1/a_1}}(R)\big)\Big)$$

where $C_{x_1 t^{1/a_1}}(R)$ is the coefficient algebra on the hypersurface $V(x_1)$.

Upper Semicontinuity:

The invariant $\mathrm{inv}_p(R)$ is upper semicontinuous in the lexicographic order. This follows directly from:

- The first component $a_1 = \mathrm{ord}_p(R)$ is upper semicontinuous.
- The closed stratum $\{q \mid \mathrm{ord}_q(R) = a_1\}$ is contained in $V(x_1)$.
- The invariant of the coefficient algebra on $V(x_1)$ is upper semicontinuous.
- By induction, inv attains its maximum on the *maximal admissible center* $V(\mathcal{A}) = V(x_1, \ldots, x_k)$.

Example: Construction of the center

$\mathcal{I}$ generated by $f = x_1^{a_1} + \ldots + x_k^{a_k}$, $a_1 \leq \ldots \leq a_k$.

- $R_1 = \mathcal{O}_X[ft] = \mathcal{O}_X[(x_1^{a_1} + \ldots + x_k^{a_k})t]$
- $\mathrm{ord}_p(R_1) = \mathrm{ord}_p(f) = a_1$
- Maximal contact:
  $x_1 t^{1/a_1} \in \mathcal{D}_X^{a_1-1}(f)t^{1/a_1} = (x_1, x_2^{a_2-a_1}, \ldots, x_k^{a_k-a_1})t^{1/a_1}$.
- $R_2 = C_{x_1 t^{1/a_1}}(R_1) = \mathcal{O}_{V(x_1)}[(x_2^{a_2} + \ldots + x_k^{a_k})t]$, as we have a unique coefficient:
  $$ft = (x_1 t^{1/a_1})^{a_1} + (x_2^{a_2} + \ldots + x_k^{a_k})t.$$
- Then $\mathrm{ord}_p(R_2) = \mathrm{ord}_p(\mathcal{O}_{V(x_1)}[(x_2^{a_2} + \ldots + x_k^{a_k})t]) = a_2$
- Maximal contact for R_2: $x_2 t^{1/a_2} \in (\mathcal{D}_{|V(x_1)}^{a_2-1}(x_2^{a_2} + \ldots + x_k^{a_k}))t^{1/a_2}$.

$$\mathcal{A}_{\mathcal{J}} = \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}], \quad \mathrm{inv}_p(\mathcal{I}) = (a_1, \ldots, a_k)$$

Admissibility and Controlled Transform

The **admissibility condition**:

$$\mathcal{I}t \subset \mathcal{A}^{\mathrm{ext}} = \mathcal{O}_X[t^{-1/w_A}, x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}],$$

translates into:

$$\mathcal{O}_B \cdot \mathcal{I}t^{w_A} \subset \mathcal{O}_B.$$

This defines the **controlled transform** of $\mathcal{I}$ under $\sigma : B \to X$:

$$\sigma^c(\mathcal{I}) = \mathcal{O}_B \cdot \mathcal{I}t^{w_A}.$$

We define the strict transform of $\mathcal{I}$ to be

$$\sigma^s(\mathcal{I}) := \{t^a f \in \mathcal{O}_B \,|\, f \in \mathcal{O}_B \cdot \mathcal{I},\ a \geq 0\} \subset \sigma^c(\mathcal{I})$$

Derivations on Cobordant Blow-ups

1. Coordinate Rescaling

On the cobordant space $B = \mathrm{Spec}_X(\mathcal{O}_X[t^{-1}, x_1 t^{w_1}, \dots, x_k t^{w_k}])$, we have the invariant coordinates $x_i' = x_i t^{w_i}$. By the chain rule, the derivations transform via a simple weight rescaling:

$$\sigma^c(\partial_{x_i}) = \partial_{x_i'} = t^{-w_i}\partial_{x_i}$$

2. The Controlled Transform of Operators

We define the controlled transform of a derivation $\sigma^c(D_X) = t^{-w_1}\sigma^{-1}(D_X) \subset \mathcal{D}_B$ such that it preserves the grading.

Key Principle: Commutativity

The action of derivations commutes with the controlled transform of functions:

$$\sigma^c(D_X\mathcal{I}) = \sigma^c(D_X)\big(\sigma^c(\mathcal{I})\big)$$

The Order of Controlled Transforms

Lemma

Let $\sigma : B = \mathrm{Spec}_X(\mathcal{O}_X[t^{-1}, x_1 t^{w_1}, \ldots, x_k t^{w_k}]) \to X$ *be a cobordant blow-up of an* R*-admissible center:*

$$\mathcal{A}^{\mathrm{ext}} = \mathcal{O}_X[t^{1/w}, \overline{x}_1 t^{1/a_1}, \ldots, \overline{x}_k t^{1/a_k}]$$

Assume $\mathrm{ord}_p(R) \leq a_1$.
Then

$$\mathrm{ord}_{p'}(\sigma^c(R)) \leq a_1 \quad \textit{for any } p' \in B.$$

Proof Sketch via Commutativity:

- Suppose $\mathrm{ord}_p(R) = a_1$. Then there exists $f \in R_b$ and a derivation D of order $N = ba_1$ such that $D^N(f) = 1$ (a unit).
- By the commutativity principle:

$$\sigma^c(D^N)\big(\sigma^c(f)\big) = \sigma^c(D^N f) = \sigma^c(1) = 1 \in \mathcal{O}_B$$

Preservation of the Maximal Contact

Lemma

Let $\sigma : B = \mathrm{Spec}_X(\mathcal{O}_X[t^{-1}, x_1 t^{w_1}, \ldots, x_k t^{w_k}]) \to X$ be a cobordant blow-up of the R-admissible center:
$\mathcal{A}^{\text{ext}} = \mathcal{O}_X[t^{-1/w}, x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]$, where $R = \mathcal{O}_X[f_j t^{b_j}]$ and x_1 is a maximal contact for R of order a_1. Then the controlled transform $\sigma^c(x_1) = x_1'$ is a maximal contact for $\sigma^c(R)$ on $\sigma^{-1}(U)$.

Proof.

We have $x_1 = D^{b_j a_1 - 1}(f_j)$.

By the commutativity of derivations and controlled transforms, we get:

$$\sigma^c(x_1) = \sigma^c\left(D^{b_j a_1 - 1} f_j\right) = \sigma^c\left(D^{b_j a_1 - 1}\right)\left(\sigma^c(f_j)\right)$$

is a maximal contact for the new algebra $\sigma^c(R)$. □

Commutativity of the Coefficient Ideal

Lemma

Let $\sigma : B \to X$ be a cobordant blow-up of an R-admissible center. The controlled transform of the algebra is generated by its elements, $\sigma^c(R) = \mathcal{O}_B[\sigma^c(f_j t^{b_j})]$, and we have the commutativity:

$$\sigma^c\left(C_{\overline{x}_1 t^{1/a_1}}(R)\right) = C_{\overline{x}'_1 t^{1/a_1}}(\sigma^c(R))$$

Proof Sketch.

- **Expansion:** Expand the generators $f_j t^{b_j}$ as polynomials in the maximal contact $\overline{x}_1 t^{1/a_1}$. The coefficients $c_{j\alpha}$ of this expansion generate the original coefficient ideal on $V(\overline{x}_1)$.
- **Transformation:** Apply σ^c. Because the transform preserves the grading and rigidly maps $\overline{x}_1 t^{1/a_1} \mapsto \overline{x}'_1 t^{1/a_1}$, the polynomial structure remains completely intact.

Cobordant Blow-ups of Maximal Centers

Proposition Let $\sigma : B \to X$ be the full cobordant blow-up of the maximal admissible center $\mathcal{A}^{\text{ext}}$ associated to the maximum invariant $\text{inv}_p(R) = (a_1, \ldots, a_k)$. Then:

1. **Preservation at the Vertex:** The maximum invariant of the controlled transform $\sigma^c(R)$ remains $(a_1, \ldots, a_k)$ and is attained precisely at the vertex $V = V(\sigma^c(\mathcal{A}))$. Furthermore, the transformed center:

$$\mathcal{A}' := \sigma^c(\mathcal{A}^{\text{ext}}) = \mathcal{O}_B[t^{-1/w}, x_1' t^{1/a_1}, \ldots, x_k' t^{1/a_k}]$$

remains maximally admissible for $\sigma^c(R)$.

2. **Strict Drop on B_+:** On the cobordant space $B_+ := B \setminus V$, the invariant strictly decreases:

$$\text{inv}(\sigma^c(R)) < (a_1, \ldots, a_k)$$

Proof Sketch

Proof Sketch via Induction on $\dim(X)$.

- $\operatorname{ord}(\sigma^c(R)) \leq a_1$.
- **Contact Preservation:** If $\overline{x}_1$ is a maximal contact for R, its transform $\overline{x}'_1 = \sigma^c(\overline{x}_1)$ is a maximal contact for $\sigma^c(R)$. The locus of the maximal order is strictly contained in $H'_1 := V(\overline{x}'_1)$ on B.
- **Commutativity:** We restrict to H'_1 and apply the commutativity $\sigma^c_{H'_1}\left(C_{\overline{x}_1 t^{1/a_1}}(R)\right) = C_{\overline{x}'_1 t^{1/a_1}}(\sigma^c(R))$
- By the inductive hypothesis on $H'_1 \subset B$, $\operatorname{inv}_p(C_{\overline{x}'_1 t^{1/a_1}}(\sigma^c(R))) \leq (a_2, \ldots, a_k)$
- The restriction $\mathcal{A}'^{\text{ext}}_{|H'_1}$ of $\mathcal{A}' := \sigma^c(\mathcal{A}^{\text{ext}})$ to H'_1 is max. admissible for $C_{\overline{x}'_1 t^{1/a_1}}(\sigma^c(R))$ $\sigma^c(\mathcal{A}) = \mathcal{O}_B[\overline{x}'_1 t^{1/a_1}, \mathcal{A}^{\text{ext}}_{B|H'_1}]$ proving the maximum invariant is precisely captured by the new vertex.

Resolution Principle

1. The invariant inv of the controlled transform:

$$\sigma^c(\mathcal{I}) := \mathcal{O}_B \cdot \mathcal{I} t^{w_A}$$

achieves its maximum at the vertex V in B, equal to its maximum along the center.

2. The invariant inv **drops** for the *cobordant blow-up* $B_+ := B \setminus V$ after removing V:

$$\max \mathrm{inv}_B(\sigma^c(\mathcal{I})) < \max \mathrm{inv}_X(\mathcal{I}),$$

leading to the **resolution of singularities**.

Example

Let $Y \subset X = \mathbb{A}^n$ be described as:

$$x_1^{b_1} + \ldots + x_n^{b_n}$$

$\mathrm{inv}_0(Y) = \max\{(a_1, \ldots, a_n) \mid (x_1^{b_1} + \ldots + x_n^{b_n})t \subset \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{int}}\}$,

$= (b_1, \ldots, b_n)$ with the center

$$\mathcal{A} = \mathcal{O}_X[x_1 t^{1/b_1}, \ldots, x_k t^{1/b_n}]^{\mathrm{int}}.$$

Rescaling gives $B = \mathrm{Spec}_X(\mathcal{O}_X[t^{-1}, x_1 t^{w_1}, \ldots, x_k t^{w_n}])$

$$\begin{aligned} \mathcal{O}_B \cdot \mathcal{I}_Y &= t^{-b_1 w_1} \left((x_1 t^{w_1})^{b_1} + \ldots + (x_n t^{w_n})^{b_n} \right) \\ &= t^{-b_1 w_1} \underbrace{\left((x_1')^{b_1} + \ldots + (x_n')^{b_n} \right)}_{\sigma^s(\mathcal{I}_Y) \text{ - strict (controlled) transform}} . \end{aligned}$$

Resolution by Removing Singularities

Strict Transform:

$$\sigma^s(\mathcal{I}_Y) = \left(\left(x_1'\right)^{b_1} + \ldots + \left(x_n'\right)^{b_n}\right),$$

has exactly the same equation as $\mathcal{I}_Y$. After **Cobordant Blow-up:**

$$\sigma_+ : B_+ \to Y.$$

The strict transform:

$$\sigma^s_+(Y) = \sigma^s(Y) \setminus \underbrace{V(x_1', \ldots, x_n')}_{\text{vertex } V}$$

on:

$$B_+ = B \setminus V,$$

becomes regular after removing the vertex V.

Resolution Process Defined by the Invariant $\mathrm{inv}_p(\mathcal{I})$

Process of Resolution:

- The maximum value of the invariant drops after each blow-up.
- The process continues until the invariant reaches the smooth point value:

$$\mathrm{inv}_p(\mathcal{I}_Y) = (1, \ldots, 1).$$

- At this stage, singularities on the strict transform of Y are resolved.

Logarithmic resolution [W22]

The definition of the invariant $\mathrm{inv}_p(\mathcal{I})$ and the maximal admissible centers can be easily adapted to the regular schemes with SNC divisor $D = \bigcup D_i$.
Consider the coordinate systems $x_1, \ldots, x_n$ where each divisorial component D_i at a given point is represented by one of the coordinates x_i.
Consider the center

$$\mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{int}}$$

We associate

- with the any free, (non-divisorial) coordinate x_i the element $b_i = a_i$ and
- with divisorial coordinate x_j the slightly "heavier" symbol $b_j = a_{j+}$.

We assume here that $a_+ > a$ for any $a \in \mathrm{R}$ and if $b > a$ for $a, b \in \mathrm{R}$ then $b > a_+$
The invariant in such a case can be written as

$$\mathrm{inv}_p(\mathcal{I}) = \max\{(b_1, \ldots, b_k) \mid \mathcal{I}t \subset \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{int}}, b_1 \leq \ldots \leq b_k\}$$

Example: The invariant in the logarithmic case

Let $f = (x_1 + x_2)^2 + x_3^7$, where x_1, x_2 are divisorial at the origin at x_3 is free. Then

$$\mathrm{inv}_0(f) = (2+, 2+, 7)$$

with the center $\mathcal{J} = \mathcal{O}_X[x_1 t^{1/2}, x_2 t^{1/2}, x_3 t^{1/7}]$. After cobordant bow-up of $\mathcal{J}$ the equation becomes the same

$$f' = ({x'}_1 + {x'}_2)^2 + {x'}_3^7$$

but we remove the vertex $V = V({x'}_1, {x'}_2, {x'}_3)$. In particular along the singular locus $V(({x'}_1 + {x'}_2), {x'}_3)$ we get

$$f' = u^2 + {x'}_3^7, \quad u := {x'}_1 + {x'}_2$$

with the smaller invariant $(2, 7) < (2+, 2+, 7)$.

Canonical logarithmic resolution

Theorem [W22]: There is a canonical logarithmic resolution $f : Y' \to Y$ by the weighted or cobordant blow-ups at the maximal admissible centers such that $f^{-1}(\mathrm{Sing}(Y))$ is a simple normal crossings (SNC) divisor on Y'. The algorithm terminates when the invariant reaches the value:

$$\mathrm{inv}_p(Y) = (1, \ldots, 1).$$

Remark: The result was established in [W22]. Recently, another algorithm was found in [ABQTW25].

From Varieties to Foliations

The method for resolving foliated varieties and integrable foliations is a natural extension of the desingularization theory using weighted centers.

Varieties as a Special Case: The orbifold (or torus action) resolution of algebraic varieties via weighted centers can be seamlessly reinterpreted as the resolution of a foliated logarithmic variety $(X, \mathcal{F})$, where the foliation is simply the full tangent sheaf:

$$\mathcal{F} = \mathcal{D}_X$$

A Unified Framework: Because of this relation, the weighted resolution machinery directly translates to the geometry of foliations, allowing us to use the exact same cobordant spaces and admissibility criteria!

Logarithmic Varieties and Derivations

- **Logarithmic Variety:** A pair (X, E), where X is a smooth irreducible variety (characteristic 0 or analytic space) and E is a Simple Normal Crossing (SNC) divisor.
- **Adapted Coordinates:** Local parameters $(x_1, \ldots, x_n)$ where any component of E is of the form $V(x_i)$.
  - x_i is *divisorial* if $V(x_i) \subset E$.
  - x_i is *free* otherwise.
- **Logarithmic Derivations ($\mathcal{D}_X^{\log}$):** The sheaf of derivations preserving E, denoted $\mathcal{D}_X(-\log E)$, locally generated by:
  - ∂_{x_i} for free coordinates.
  - $x_j\partial_{x_j}$ for divisorial coordinates.

Distributions and Foliations

- **Distribution:** A coherent subsheaf $\Delta \subset \mathcal{D}_X$ (or $\Delta \subset \mathcal{D}_X^{\log}$ for logarithmic distributions).

- **Foliation ($\mathcal{F}$):** An *involutive* distribution. That is, it is closed under the Lie bracket. For any local vector fields $X^1, X^2 \in \mathcal{F}(U)$:

$$[X^1, X^2] = X^1X^2 - X^2X^1 \in \mathcal{F}(U)$$

- **Foliated Logarithmic Variety:** A triple $(X, \mathcal{F}, E)$, where X is smooth, E is SNC, and $\mathcal{F} \subset \mathcal{D}_X^{\log}$ is a foliation.

Singularities of Foliations

- **Rank:** The rank of $\mathcal{F}$ as a coherent sheaf.
- **Singular Set ($\mathbf{Sing}(\mathcal{F})$):** The set of points $\mathfrak{a} \in X$ where the quotient $\mathcal{D}_{X,\mathfrak{a}}/\mathcal{F}_{\mathfrak{a}}$ (or the logarithmic analogue) is *not* locally free.
- **Nonsingular Points:** If $\mathcal{D}_{X,\mathfrak{a}}/\mathcal{F}_{\mathfrak{a}}$ is locally free, $\mathcal{F}$ is *nonsingular* at $\mathfrak{a}$. The exact sequence splits:

$$0 \to \mathcal{F}_{\mathfrak{a}} \to \mathcal{D}_{X,\mathfrak{a}} \to \mathcal{D}_{X,\mathfrak{a}}/\mathcal{F}_{\mathfrak{a}} \to 0$$

  making both $\mathcal{F}_{\mathfrak{a}}$ and the quotient free $\mathcal{O}_{X,\mathfrak{a}}$-submodules.
- **Nonsingular Locus:** Denoted $(X, \mathcal{F})^{\text{nsing}} = X \setminus \operatorname{Sing}(\mathcal{F})$.

Pfaffian Presentation and Saturation

- **Logarithmic Forms ($\Omega_X^{\log}$):** The dual sheaf of $\mathcal{D}_X^{\log}$, generated locally by the forms dx_i (free) and $\frac{dx_j}{x_j}$ (divisorial).
- **Conormal Sheaf ($N^*_{\mathcal{F}}$):** The kernel of the canonical map $\Omega_X^{\log} \to \mathrm{Hom}(\mathcal{F}, \mathcal{O}_X)$. We denote this by $N^*_{\mathcal{F}} = \mathcal{F}^{\perp}$. The fact that $\mathcal{F}$ is closed under Lie bracket operations implies:

$$d\omega \wedge \omega_1 \wedge \cdots \wedge \omega_d \equiv 0 \quad (\text{where } d = n - \mathrm{rank}(\mathcal{F}))$$

  for all $\omega, \omega_1, \ldots, \omega_d \in \mathcal{F}^{\perp}$.
- **Saturation:** The saturation of $\mathcal{F}$ is defined as $\mathcal{F}^{\mathrm{sat}} := \mathcal{F}^{\perp\perp}$, which is a coherent sub-sheaf of $\mathcal{D}_X^{\log}$.
- **Saturated Foliation:** We say $\mathcal{F}$ is *saturated* if $\mathcal{F} = \mathcal{F}^{\mathrm{sat}}$. If $\mathcal{F}$ is smooth at a point a, it is locally a direct summand, meaning it is saturated over its smooth locus U. Thus we have:

$$\mathcal{F}^{\mathrm{sat}} = \{\partial \in \mathcal{D}_X^{\log} \mid \partial_{|U} \in \mathcal{F}_{|U}\}$$

Pullbacks and Inverse Images

Let $\phi : (X, E) \to (Y, F)$ be a morphism. A saturated foliation $\mathcal{G} \subset \mathcal{D}_Y^{\log}$ is naturally defined by its conormal sheaf $N_{\mathcal{G}}^* := \mathcal{G}^\perp$.

1. Pullback of Forms:
The natural pullback $d\phi^*$ maps $\Omega_Y^{\log} \to \Omega_X^{\log}$, acting locally as $\omega \mapsto \omega \circ d\phi$.

2. Inverse Image:
The *inverse image* $\phi^{-1}(\mathcal{G})$ is the saturated foliation in $\mathcal{D}_X^{\log}$ that is exactly dual to the pulled-back forms:

$$\phi^{-1}(\mathcal{G}) := \left(d\phi^*(N_{\mathcal{G}}^*)\right)^\perp \subset \mathcal{D}_X^{\log}$$

3. Restriction:
If $i : X \to Y$ is a closed embedding, the inverse image $i^{-1}(\mathcal{G})$ is called the *restriction*, denoted $\mathcal{G}_{|X}$.

Transforms of Derivations

Let $\phi : X \to Y$ be a proper birational map with exceptional divisor D. A derivation $\partial \in \mathcal{D}_Y^{\log}$ pulls back to a unique meromorphic *total transform* $\phi^*(\partial) \in \mathcal{D}_X^{\log}(*D)$.

Let y_i be local defining sections for the components of $D = \sum_{i=1}^k D_i$. We define two regularized transforms in $\mathcal{D}_X^{\log}$ by multiplying by the minimal necessary powers to clear poles:

1. **Controlled Transform (Non-negative shifts):**

$$\phi^c(\partial) := y_1^{r_1} \cdots y_k^{r_k} \phi^*(\partial) \quad \text{for minimal } r \in \mathbb{Z}_{\geq 0}^k$$

2. **Strict Transform (Integer shifts):**

$$\phi^s(\partial) := y_1^{m_1} \cdots y_k^{m_k} \phi^*(\partial) \quad \text{for minimal } m \in \mathbb{Z}^k$$

Remark: These transforms are locally defined up to a unit.

Transforms of Foliations

Using the transformed derivations, we define the corresponding transforms for a foliation $\mathcal{G} \subset \mathcal{D}_Y^{\log}$.

- **Total Transform $\phi^*(\mathcal{G})$:** The $\mathcal{O}_X$-subsheaf of $\mathcal{D}_X^{\log} \otimes K(X)$ generated by the total transforms $\phi^*(\partial)$ for all $\partial \in \mathcal{D}_Y^{\log}$.
  - *Warning:* This typically introduces rational derivations with poles!
- **Controlled Transform $\phi^c(\mathcal{G})$:** The $\mathcal{O}_X$-subsheaf of $\mathcal{D}_X^{\log}$ generated by the controlled transforms $\phi^c(\partial)$.
- **Strict Transform $\phi^s(\mathcal{G})$:** The $\mathcal{O}_X$-subsheaf of $\mathcal{D}_X^{\log}$ generated by the strict transforms $\phi^s(\partial)$.

Structural Properties and Saturations

We can alternativey describe these transforms using logarithmic structures and saturation:

- **Controlled via Intersection:**

$$\phi^c(\mathcal{G}) = \phi^*(\mathcal{G}) \cap \mathcal{D}^{\log}_{(X,D)}$$

  (Also known as the *nonsaturated inverse transform* or *strict analytic transform* in some literature when matching E).

- **Strict via Saturation:** The strict transform $\phi^s(\mathcal{G})$ is the $\mathcal{D}^{\log}_{(X,D)}$-saturation of $\phi^c(\mathcal{G})$.
  - If $\mathcal{G}$ is saturated on (Y, F), then $\phi^s(\mathcal{G}) = \phi^{-1}(\mathcal{G})$ is simply the saturated inverse image of $\mathcal{G}$.

Rationally Totally Integrable Foliations

Definition (Totally Integrable Foliations)

Let $(X, \mathcal{F}, E)$ be a smooth foliated logarithmic variety. We say that $\mathcal{F}$ is *globally rationally (or meromorphically) totally integrable* if there exists a rational (or meromorphic) morphism $\varphi : X \dashrightarrow B$ such that:

$$\mathcal{F} = \mathcal{D}^{\log}_{X/B}, \quad \mathcal{F}{\cdot}\mathcal{O}_{X,p} = \mathrm{span}_{\mathcal{O}_{X,p}} \left\{ \partial \in \mathcal{D}^{\log}_{X};\ \partial(f \circ \varphi) \equiv 0,\ \forall f \in \mathcal{O}_{B,\varphi(p)} \right\}.$$

$\mathcal{K}$-Monomial Foliations

Definition (Belotto)

Let $(X, \mathcal{F}, E)$ be a foliated logarithmic variety over $\mathbb{K}$, and let $\mathcal{K} \subset \mathbb{K}$ be a field. A foliation $\mathcal{F}$ is said to be *$\mathcal{K}$-monomial* at a point $\mathfrak{a}$ if there exists a regular coordinate system (w, v) in a neighborhood U of $\mathfrak{a}$ such that $\mathcal{F}$ can be locally generated on U by:

$$\partial_{v_i} \quad \text{and} \quad \nabla_j = \sum_{k=1}^{n} a_{jk} w_k \, \partial_{w_k},$$

where $j = 1, \ldots, n$, $i = 1, \ldots, r$, and $a_{jk} \in \mathcal{K}$. Alternatively, in terms of regular forms, $\mathcal{F}$ can also be expressed as:

$$\prod_{j=1}^{n} w_j \cdot \sum_{j=1}^{n} \beta_{ij} \frac{dw_j}{w_j}(\partial) \equiv 0, \quad i = 1, \ldots, r,$$

where $\beta_{ij} \in \mathcal{K}$.

Globally $\mathcal{K}$-Darboux Totally Integrable Foliations

Definition (Globally $\mathcal{K}$-Darboux Totally Integrable Foliations)

Let $(X, \mathcal{F}, E)$ be a foliated logarithmic variety over $\mathbb{K}$ and $\mathcal{K} \subset \mathbb{K}$ be a field. We say that the foliation $\mathcal{F}$ is *globally $\mathcal{K}$-Darboux totally integrable* if:

- There exists a rational (or meromorphic) morphism $\varphi : X \dashrightarrow B$ whose graph projects properly over X.
- There exists a $\mathcal{K}$-monomial foliation $\mathcal{G}$ over a smooth variety B.
- The foliation $\mathcal{F}$ is the inverse transform of $\mathcal{G}$ by φ.

Presentation of Darboux First Integrals

1. Classical Description

A foliation $\mathcal{F}$ admits global $\mathcal{K}$-Darboux first integrals if it is defined by the vanishing of logarithmic forms:

$$\sum_{j=1}^{n_i} \beta_{ij} \frac{df_{ij}}{f_{ij}}(\partial) \equiv 0 \quad \text{for } i = 1, \ldots, r$$

where $f_{ij} \in K(X)$ and $\beta_{ij} \in \mathcal{K}^*$.

- **The Morphism:** The functions f_{ij} naturally define a rational map to a product of projective spaces:

$$\phi : X \dashrightarrow \prod_{i=0}^{r} \mathbb{P}^{n_i}$$

- **The Target Foliation:** The constants β_{ij} canonically determine a $\mathcal{K}$-monomial foliation $\mathcal{G}$ on the target space. $\sum_{j=1}^{n_i} \beta_{ij} \frac{dx_{ij}}{x_{ij}}(\partial) \equiv 0 \quad \text{for } i = 1, \ldots, r$

The Classical Rectification Theorem

Theorem (Flow Box Theorem)

Suppose X is a coherent analytic space. Let $\partial \in \mathcal{D}_{X,\mathfrak{a}}$ and $x \in \mathcal{O}_{X,\mathfrak{a}}$ be such that $\partial(x)$ is a unit. Then there exists a Euclidean coordinate system $(x, y) = (x, y_1, \dots, y_{n-1})$ centered at $\mathfrak{a}$ such that:

$$\partial = V(x,y)\partial_x \sim \partial_x$$

where $V(x,y)$ is a unit.

The Algebraic Obstruction:

This result plays a fundamental role in the local theory of foliations. It is **impossible** to obtain a rectification theorem over strictly *regular* (algebraic) or *étale* coordinate systems.

The Need for Nested-Regular Coordinates

Example

Consider the derivation $\partial = \partial_x - y\partial_y$.

- **Analytic Rectification:** By defining the first integral $y' := e^x y$, we have $\partial(y') = 0$. Thus, with respect to the Euclidean coordinate system (x, y'), the derivation straightens to exactly $\partial = \partial_x$.
- **The Catch:** Because e^x is transcendental, there is no regular or étale coordinate system $(x, \bar{y})$ capable of rectifying ∂.

Why Formal is Not Enough:
A purely formal version of the rectification theorem destroys the geometric structure required to prove the semi-continuity of the invariant $\mathrm{inv}_{\mathcal{F}}$.

The Solution: We must bridge this gap by proving a **Nested-Regular Rectification Theorem**.

Formal Completions and Algebraizable Ideals

To define regular blow-up centers, our coordinates must satisfy strict coherency properties- they cannot be purely formal. We introduce an intermediate class of coordinates.

Setup:

- Let $I \subset \mathcal{O}_{X,\mathfrak{a}}$ be a regular ideal.
- We denote by
$$\widehat{\mathcal{O}}_{X,I} := \varprojlim_n \mathcal{O}_{X,\mathfrak{a}}/I^n$$
  the I-adic completion of $\mathcal{O}_{X,\mathfrak{a}}$.

Convention (Algebraizable Ideals):
We say that a formal ideal $I \subset \widehat{\mathcal{O}}_{X,\mathfrak{a}}$ is **algebraizable** if its restriction to the regular local ring generates the formal ideal. That is, if $I' := I \cap \mathcal{O}_{X,\mathfrak{a}}$, then:
$$I' \cdot \widehat{\mathcal{O}}_{X,\mathfrak{a}} = I$$

Nested-Regular Coordinates

Definition (Nested-Regular Coordinates)

A partial local coordinate system $(x_1, \dots, x_k) \in \widehat{\mathcal{O}}_{X,\mathfrak{a}}$ at a point $\mathfrak{a}$ is called **nested-regular** if:

1. The first coordinate is fully regular: $x_1 \in \mathcal{O}_{X,\mathfrak{a}}$.
2. For each $i = 1, \dots, k$, the formal ideal $I_i = (x_1, \dots, x_i)$ is algebraizable.
3. The subsequent coordinate belongs to the successive completion: $x_{i+1} \in \widehat{\mathcal{O}}_{X,I_i}$.

Notation:
Given a partial nested-regular coordinate system $x = (x_1, \dots, x_k)$, we denote by $\widehat{\mathcal{O}}_{X,(x)}$ the ring $\widehat{\mathcal{O}}_{X,I}$, where $I = (x_1, \dots, x_k)$ is the algebraizable ideal.

The Nested-Regular Rectification Theorem

Theorem: Let (X, E) be a logarithmic variety and $\partial \in \mathcal{D}^{\log}_{X,\mathfrak{a}}$. Let $x \in \mathcal{O}_{X,\mathfrak{a}}$ be a germ such that $\partial(x)$ is a unit, and set $H := V(x)$. There exists a nested-regular coordinate system $(x, \widehat{y}) = (x, \widehat{y}_1, \dots, \widehat{y}_{n-1}) \in \widehat{\mathcal{O}}_{X,(x)}$ adapted to E such that:

$$\partial = V(x, \widehat{y})\partial_x \sim \partial_x$$

where V is a unit, and the restrictions $y_i = \widehat{y}_{i|H} \in \mathcal{O}_{H,\mathfrak{a}}$ are fully regular.

Key Consequence:

There exists a lifting $\ell_{x,\partial} : \mathcal{O}_{H,\mathfrak{a}} \to \widehat{\mathcal{O}}_{X,(x)}$ given by $\ell_{x,\partial}(y_i) = \widehat{y}_i$, such that the foliation derivations annihilate the lifted variables:

$$\partial(\ell_{x,\partial}(f)) \equiv 0 \quad \text{for all } f \in \mathcal{O}_{H,\mathfrak{a}}$$

Proof Sketch: Constructing the Invariant Coordinates

Assume $\partial(x) \equiv 1$. For any initial regular coordinate $y \in \mathcal{O}_{X,\mathfrak{a}}$, we iteratively construct a sequence $(y_d)_{d\in\mathbb{N}}$ where $y_0 = y$, defined by:

$$y_d := y_0 - \sum_{j=1}^{d} \frac{x^j}{j!}\partial^j(y_0)$$

This sequence satisfies crucial properties:

1. $y_c - y_d \in (x^{c+1})$ for $c \leq d$ (convergence in the (x)-adic topology).
2. $\partial(y_d) = \frac{x^d}{d!}\partial^{d+1}(y_0) \in (x^d)$ (the derivative vanishes in the limit).
3. If y is divisorial, then $y_d \in (y_0)$ (preserves boundary adaptation).

The formal limit $\widehat{y} = \lim y_d \in \widehat{\mathcal{O}}_{X,(x)}$ strictly satisfies $\partial(\widehat{y}) \equiv 0$, yielding the desired nested-regular invariant coordinates.

Example: Nested Rectification

Consider the following derivation:

$$\partial = \partial_x - y\partial_y$$

The Coordinate Change:
We introduce a new transverse coordinate $y' := e^x y$.

$$\partial(y') = \partial_x(e^x y) - y\partial_y(e^x y) = e^x y - e^x y = 0$$

The Nested Rectified System:
While y' is not algebraic we get

$$y' = e^x y \in \widehat{\mathcal{O}}_{X,(x)}$$

In this *nested coordinate* system (x, y'),

$$\partial = \partial_x$$

Note that $y'_{|V(x)} = y_{|V(x)}$ is algebraic implying $(x, y') = (x, y)$ describes a regular algebraic center on $V(x)$ and on X.

Liftings Along a Transverse Coordinate

Let $x \in \mathcal{O}_{X,\mathfrak{a}}$ be a coordinate and set

$$H := V(x).$$

Lifting

A *lifting* is an injective homomorphism

$$\ell : \mathcal{O}_{H,\mathfrak{a}} = \mathcal{O}_{X,\mathfrak{a}}/(x) \longrightarrow \widehat{\mathcal{O}}_{X,(x)}$$

which is a right inverse of the natural projection

$$\widehat{\mathcal{O}}_{X,(x)} \twoheadrightarrow \mathcal{O}_{H,\mathfrak{a}}.$$

Liftings Associated to (x, ∂)

Associated lifting

Assume

$$\partial(x) \in \mathcal{O}_{X,\mathfrak{a}}^{\times}, \qquad H = V(x).$$

A lifting

$$\ell : \mathcal{O}_{H,\mathfrak{a}} \to \widehat{\mathcal{O}}_{X,(x)}$$

is associated to (x, ∂) if

$$\ell(\mathcal{O}_H) = \left(\widehat{\mathcal{O}}_{X,(x)}\right)^{\partial} := \{f : \partial(f) = 0\}.$$

Denote it by $\ell_{x,\partial}$.

If (x, y) are nested coordinates with $\partial = \partial_x$, then

$$\widehat{\mathcal{O}}_{X,(x)} = \mathcal{O}_H[[x]], \qquad \mathcal{O}_H \cong \ell_{x,\partial}(\mathcal{O}_H) = \left(\widehat{\mathcal{O}}_{X,(x)}\right)^{\partial},$$

i.e. ∂-invariants depend only on y.

Local Splitting Along Transverse Hypersurfaces

Theorem (Local Splitting)

Let $(X, \mathcal{F}, E)$ be a foliated logarithmic variety and suppose

$$x \in \mathcal{O}_{X,\mathfrak{a}}, \qquad \partial \in \mathcal{F}_{\mathfrak{a}}, \qquad \partial(x) \in \mathcal{O}_{X,\mathfrak{a}}^{\times}.$$

Set

$$H = V(x), \qquad \widehat{\mathcal{F}}_{(x)} = \widehat{\mathcal{O}}_{X,(x)} \cdot \mathcal{F}.$$

Then there exist nested regular coordinates (x, y) such that

$$\partial = u\,\partial_x, \qquad u \in \widehat{\mathcal{O}}_{X,(x)}^{\times}, \quad \textit{and}$$

$$\widehat{\mathcal{F}}_{(x)} = \operatorname{span}_{\widehat{\mathcal{O}}_{X,(x)}}\Big(\partial_x,\, d\ell_{x,\partial_x}(\mathcal{F}_{|H})\Big) = \operatorname{span}_{\widehat{\mathcal{O}}_{X,(x)}}\Big(\partial_x,\, \mathcal{F}_{|H}\Big).$$

Sketch of the Proof

Assume $\partial = \partial_x$ in nested coordinates $(x, y) = (x, y_1, \ldots, y_k)$. For

$$D = A(x, y)\partial_x + \sum_i B_i(x, y)\partial_{y_i} \in \widehat{\mathcal{F}}_{(x)}, \quad \text{let}$$

$D_H := \sum_i B_i(x, y)\partial_{y_i}$.
Its restriction to $H = V(x)$ is

$$D_{|H} = \sum_i B_i(0, y)\partial_{y_i}. \quad \text{Then}$$

$$[D_H, \partial_x] = \sum_i \partial_x(B_i)\partial_{y_i} \in \widehat{\mathcal{F}}_{(x)},$$

$$D_1 := D - x[D_H, \partial_x] \equiv D_{|H} \pmod{(x)}.$$

Inductively construct

$$D_i \in \widehat{\mathcal{F}}_{(x)}, \qquad D_i \equiv D_{|H} \pmod{(x^i)}.$$

Hence the lifting of $D_{|H}$ lies in $\widehat{\mathcal{F}}_{(x)}$.

Foliations and $\mathcal{F}$-Aligned Centers

Definition Let $(X, E, \mathcal{F})$ be a foliated logarithmic variety. An adapted center $\mathcal{A}_J$ is $\mathcal{F}$-**aligned** if

- **Formal Presentation:** There exist nested-regular coordinates (x, y) at p adapted to E such that:

$$\mathcal{A}_J \cdot \widehat{\mathcal{O}}_{X,(x)} = \widehat{\mathcal{O}}_{X,(x)}[x_i t^{1/a_i}, y_j t^{1/b_j}]^{\text{int}}$$

  where x_i are free variables.
- **Foliation Splitting:** The foliation splits as:

$$\widehat{\mathcal{F}}_{(x)} = \operatorname{span}(\partial_{x_1}, \ldots, \partial_{x_l}, \nabla_1, \ldots, \nabla_m)$$

  where the derivations ∇_j are independent of (x_i, ∂_{x_i}), and the (y)-part of the Rees algebra is strictly $\widehat{\mathcal{F}}_p$-invariant:

Invariance Condition

$$\widehat{\mathcal{F}}_{(x)} \left(\widehat{\mathcal{O}}_{X,(x)}[y_j t^{1/b_j}]^{\text{int}} \right) \subset \widehat{\mathcal{O}}_{X,(x)}[y_j t^{1/b_j}]^{\text{int}}$$

Definition

Let $\mathcal{I}$ be an ideal (resp. R is a Rees algebra) on $(X, E, \mathcal{F})$ be a foliated log. variety. We say that $\mathcal{A}_{\mathcal{J}}$ is $(\mathcal{I}, \mathcal{F})$-admissible if there exists $\mathcal{F}$-aligned center $\mathcal{A}_J$ such that

$$\mathcal{I}t \text{ (resp.} R) \subset \widehat{\mathcal{O}}_{X,p}[x_i t^{1/a_i}, y_j t^{1/b_j}]^{\mathrm{int}}$$

$(\mathcal{I}, \mathcal{F})$-**Admissibility and the Classical Case** $\mathcal{F} = \mathcal{D}_X$**,** $E = \emptyset$
Because $\mathcal{D}_X$ has maximal rank everywhere, **all** coordinates are purely transverse,the $\mathcal{F}$-invariant variables (y) are absent. The presentation of the center collapses :

$$\mathcal{A}_J = \mathcal{O}_X[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}]^{\mathrm{Int}}$$

The Canonical Invariant $\mathrm{inv}_{\mathcal{F}}$

Definition: Let $\mathcal{I}$ be an ideal (resp. R is a Rees algebra) on $(X, E, \mathcal{F})$ be a foliated log. variety and $p \in X$. The canonical invariant $\mathrm{inv}_{p,\mathcal{F}}(\mathcal{I})$ is defined as the maximum weight sequence:

$$\max \Big\{ (a_1, \ldots, a_k, \infty + b_1, \ldots, \infty + b_r) \mid \mathcal{I}t \ (\text{resp.} R) \subset \widehat{\mathcal{O}}_{X,p}[x_i t^{1/a_i}, y_j t^{1/b_j}]^{\mathrm{int}} \Big\}$$

where the maximum is taken over all $(\mathcal{I}, \mathcal{F})$-**admissible** presentations. Specifically, the coordinates (x, y, z) form an $\mathcal{F}$-**aligned presentation** $\mathcal{A}_J$:

- **Transverse part:** Controlled by x_i, where the foliation splits as $\widehat{\mathcal{F}}_{(x)} = \mathrm{span}(\partial_{x_1}, \ldots, \partial_{x_k}, \nabla_1, \ldots, \nabla_m)$.
- **Invariant part:** Formed by the free variables y_j and E-divisorial variables z_l, satisfying the strict invariance condition:

$$\widehat{\mathcal{F}}_{(x)} \left(\widehat{\mathcal{O}}_{X,p}[y_j t^{1/b_j}]^{\mathrm{int}} \right) \subset \widehat{\mathcal{O}}_{X,p}[y_j t^{1/b_j}]^{\mathrm{int}}$$

$\mathcal{F}$-Order and Rational Rees Algebras

Definition: The $\mathcal{F}$*-order* of an ideal $\mathcal{I}$ at a point $p \in X$ with respect to a foliation $\mathcal{F}$ is given by:

$$\operatorname{ord}_{\mathcal{F},p}(\mathcal{I}) := \min\{n \in \mathbb{Z}_{\geq 0} \mid \mathcal{F}^n(\mathcal{I})_p = \mathcal{O}_{X,p}\} \cup \{\infty\}.$$

Similarly for $R = \bigoplus R_a$ we have

$$\operatorname{ord}_{\mathcal{F},p}(R) := \min\{\operatorname{ord}_{\mathcal{F},p}(R_a)/a \mid a > 0\}.$$

In particular for $\mathcal{F}$-aligned center $\mathcal{A}_{\mathcal{J}} = \widehat{\mathcal{O}}_{X,p}[x_i t^{1/a_i}, y_j t^{1/b_j}]^{\mathrm{int}}$ we get

that $\operatorname{ord}_{\mathcal{F},p} = a_1$, which implies that if
$\operatorname{inv}_{p,\mathcal{F}}(\mathcal{I}) := \max\{(a_1, \ldots, a_k, \infty + b_1, \ldots, \infty + b_r,)$ then

$$\operatorname{ord}_{p,\mathcal{F}}(\mathcal{I}) = a_1$$

$\mathcal{F}$-Maximal Contact and the $\mathcal{F}$-Order

Consider the formal admissibility relation

$$\mathcal{I}t^a \subset \widehat{\mathcal{O}}_{X,p}[m_p t],$$

where $a = \operatorname{ord}_{\mathcal{F},p}(\mathcal{I})$. Acting by the foliation derivations

$t^{-1}\mathcal{F} = \mathrm{span}(t^{-1}\partial)$ we enlarge the left side and preserve the right side $\widehat{\mathcal{O}}_{X,p}[m_p t]$. In gradation 1 we obtain:

$$\mathcal{F}^{a-1}(\mathcal{I})t \subset m_p t.$$

Since $\mathcal{F}^a(\mathcal{I})_p = \mathcal{O}_{X,p}$ by the definition of the $\mathcal{F}$-order, there exists a coordinate $x_1 \in \mathcal{F}^{a-1}(\mathcal{I})$ such that $\mathcal{F}(x_1)_p = \mathcal{O}_{X,p}$. This x_1 is purely

transverse to the foliation and is called an $\mathcal{F}$-**maximal contact** with $\partial_{x_1} \in \mathcal{F}$.

$\mathcal{F}$-Maximal Contact for Admissibility Relation

Rescaling the procedure, we construct the $\mathcal{F}$-aligned center. Consider the **maximal** $(\mathcal{I}, \mathcal{F})$-**admissibility relation**:

$$\mathcal{I}t \subset \widehat{\mathcal{A}}_{\mathcal{J}} = \widehat{\mathcal{O}}_{X,p}[x_1 t^{1/a_1}, \ldots, x_k t^{1/a_k}, y_j t^{1/b_j}]^{\text{int}}.$$

By maximality, $a_1 = \operatorname{ord}_{\mathcal{F},p}(\mathcal{I}) \leq a_2 \leq \ldots \leq a_k$. Acting by the rescaled

derivations $t^{-1/a_1}\widehat{\mathcal{F}}_{(x)}$ on both sides, we preserve the right side (because the y, z part of the Rees algebra is strictly $\widehat{\mathcal{F}}_{(x)}$-invariant) and enlarge the left side. As before, we obtain the $\mathcal{F}$-maximal contact coordinate:

- $x_1 t^{1/a_1} \in \mathcal{F}^{a_1 - 1}(\mathcal{I}) t^{1/a_1} \subset \widehat{\mathcal{A}}_{\mathcal{J}}$.
- $\partial_{x_1} \in \mathcal{F}$

Thus we have isolated the first $\mathcal{F}$-**transverse** coordinate of the maximal $\mathcal{F}$-aligned center!

Reduction Property of the Coefficient Algebra

- Let $\mathcal{R}$ be a Rees algebra of $\mathcal{F}$-order a_1 at a point $p \in X$, with $\mathcal{F}$-maximal contact given by $x_1 t^{1/a_1}$.
- Let $\widehat{\mathcal{A}}_{\mathcal{J}}$ be the maximal $\mathcal{F}$-aligned center for $\mathcal{R}$ at p.

Then the resolution problem drops in dimension:

- The restriction $\widehat{\mathcal{A}}_{\mathcal{J}|H}$ is exactly the maximal $\mathcal{F}_{|H}$-aligned center for the coefficient $\mathcal{C}_{x_1 t^{1/a_1}}(R)$ on H.
- $\widehat{\mathcal{A}}_{\mathcal{J}} = \widehat{\mathcal{O}}_{X,p}\left[x_1 t^{1/a_1}, \ell(\widehat{\mathcal{A}}_{\mathcal{J}|H})\right]^{\text{int}}$.
- $\widehat{\mathcal{F}} = \widehat{\mathcal{O}_{X,x}}(\partial_{x_1}, \mathcal{F}_{|V(x_1)})$

This formal splitting perfectly decouples the $\mathcal{F}$-transverse geometry from the $\mathcal{F}$-invariant geometry recursively!

Construction of the Maximal $\mathcal{F}$-Aligned Center

The construction of the maximal $\mathcal{F}$-aligned center for $\mathcal{I}$ at p isolates the transverse coordinates recursively. Over the formal completion, define:

$$\mathcal{R}_1 := \mathcal{O}_{X,p}[\mathcal{I}t] \subset \mathcal{A}_1 := \mathcal{A}_{\mathcal{J}}$$

and for $i = 1, \ldots, k$:

$$\mathcal{R}_{i+1} := \mathcal{C}_{x_i t^{1/a_i}}(\mathcal{R}_i) \subset \mathcal{A}_{i+1} := \mathcal{A}_{i|V(x_i)}$$

$$\widehat{\mathcal{A}}_{\mathcal{J}} = \widehat{\mathcal{O}}_{X,p}\left[x_1 t^{1/a_1}, \ldots, x_i t^{1/a_i}, \ell(\mathcal{A}_{i+1})\right]^{\text{int}}$$

The transverse reduction ends when $\mathrm{ord}_{\mathcal{F},p}(\mathcal{R}_{k+1}) = \infty$, yielding the strictly $\mathcal{F}$-invariant part:

$$\mathcal{A}_{k+1} = \mathcal{O}_{X,p}[y_j t^{1/b_j}]^{\text{int}}$$

where $(x, y) = (x_1, \ldots, x_k, y_1, \ldots, y_{n-k})$.
$\widehat{\mathcal{F}}_{(x)} = \widehat{\mathcal{O}_{X,x}}(\partial_{x_1}, \ldots, \partial_{x_k}, \mathcal{F}_{|V(x)}) = \widehat{\mathcal{O}_{X,x}}(\partial_{x_1}, \ldots, \partial_{x_k}, \nabla_1(y), \ldots, \nabla_k(y))$

At each step:

- $a_i := \operatorname{ord}_{\mathcal{F},p}(\mathcal{R}_i)$
- $x_i t^{1/a_i}$ is an $\mathcal{F}$-**maximal contact** element for $\widehat{\mathcal{R}}_i$.
- $\mathcal{R}_{i+1} = \mathcal{C}_{x_i t^{1/a_i}}(\mathcal{R}_i) \subset \mathcal{A}_i = \mathcal{A}_{|V(x_1,\ldots,x_{i-1})}$ is maximal admissible
- $\widehat{\mathcal{F}}_{(x)} = \widehat{\mathcal{O}_{X,x}}(\partial_{x_1},\ldots,\partial_{x_i},\mathcal{F}_{|V(x_1,\ldots,x_i)})$

At the end $\operatorname{ord}_{\mathcal{F},p}(\mathcal{R}_{k+1}) = \infty$, with
$\mathcal{F}$-admissibility relation:

$$\mathcal{R}_{k+1} \subset \mathcal{A}_{k+1} = \mathcal{O}_{X,p}[y_j t^{1/b_j}]^{\text{int}}$$

Invariant Part: $\mathrm{ord}_{\mathcal{F},p}(\mathcal{R}) = \infty$

Assume the transverse reduction terminates at step $k+1$:

$$\mathrm{ord}_{\mathcal{F},p}(\mathcal{R}_{k+1}) = \infty.$$

The $\mathcal{F}$-admissibility relation gives

$$\mathcal{R}_{k+1} \subset \mathcal{A}_{k+1} = \mathcal{O}_{X,p}[y_j t^{1/b_j}]^{\mathrm{int}}.$$

Step 1: $\mathcal{F}$-Saturation

Since $\mathcal{A}_{k+1}$ is $\mathcal{F}$-invariant, admissibility is preserved under saturation:

$$\mathcal{F}^{\infty}(\mathcal{R}_{k+1}) \subset \mathcal{A}_{k+1}.$$

Step 2: Final Extraction

In the classical setting,

$$\mathcal{A}_{k+1} = \mathcal{O}_{V(x),p}[y_j t^{1/b_j}]^{\mathrm{int}}$$

is the maximal admissible center for

$$\mathcal{F}^{\infty}(\mathcal{R}_{k+1}), \qquad \mathrm{inv}_p\big(\mathcal{F}^{\infty}(\mathcal{R}_{k+1})\big) = (b_1, \dots, b_\ell).$$

Construction of the Maximal $\mathcal{F}$-Aligned Center

Canonicity and compatibility with formal automorphisms imply that the maximal admissible center

$$\mathcal{A}_{k+1} = \mathcal{O}_{V(x)}[y_j t^{1/b_j}]^{\mathrm{int}}$$

for

$$\mathcal{F}^{\infty}(\mathcal{R}_{k+1})$$

is $\mathcal{F}$-invariant, hence $\mathcal{F}$-aligned.

Thus we obtain the maximal $\mathcal{F}$-aligned center for R, yielding

$$\mathrm{inv}_{p,\mathcal{F}}(R) = (a_1, \ldots, a_k, \infty + b_1, \ldots, \infty + b_\ell).$$

Split Transforms by Cobordant Blow-ups

A cobordant blow-up $\sigma : B \to X$ with center $\mathcal{A}_J$ possesses a canonical splitting given by the natural birational morphism:

$$B \to X \times \mathbb{A}^1 = \mathrm{Spec}_X(\mathcal{O}_X[t^{-1}])$$

This splitting determines canonical transforms for derivations.

Definition (Split Total Transform of a Derivation)

Let ∂ be a rational derivation over $K(X)$. The **total transform** of ∂ by a full cobordant blow-up σ is a derivation $\sigma^*(\partial)$ over $K(B)$ uniquely determined by:

$$\sigma^*(\partial)(t_B^{-1}) \equiv 0 \quad \text{and} \quad \sigma^*(\partial)\sigma^*(f) = \sigma^*(\partial(f)), \quad \forall f \in \mathcal{O}_X$$

Controlled and Strict Transforms of Derivations

Given $\partial \in \mathcal{D}_X^{\log}$, its total transform $\sigma^*(\partial)$ is regular over B_-. Multiplying by a sufficiently large power of t_B^{-1} yields a regular derivation in $\mathcal{D}_B^{\log}$.

Definition (Controlled and Strict Transforms under cobordant blow-up)

Given a derivation $\partial \in \mathcal{D}_X^{\log}$, we define *its controlled transform* to be :

$$\sigma^c(\partial) = t_B^{-a_\partial} \sigma^*(\partial) \in \mathcal{D}_B^{\log}$$

$$\text{where } a_\partial := \min\{a \in \mathbb{Z}_{\geq 0} \mid t_B^{-a} \sigma^*(\partial) \in \mathcal{D}_B^{\log}\}$$

$$\sigma^s(\partial) = t_B^{-b_\partial} \sigma^*(\partial) \in \mathcal{D}_B^{\log}, \quad \textit{its strict transform}:$$

$$\text{where } b_\partial := \min\{b \in \mathbb{Z} \mid t_B^{-b} \sigma^*(\partial) \in \mathcal{D}_B^{\log}\}$$

Note: The main difference is that the strict transform allows negative powers ($b \in \mathbb{Z}$), whereas the controlled transform restricts to non-negative shifts.

Local Coordinates

Recall that the full cobordant blow-up

$$B = \mathrm{Spec}_X(\mathcal{O}_X[t^{-1}, x_1 t^{w_1}, \ldots, x_k t^{w_k}]) \to X$$

is described by one chart with coordinates:

- t^{-1};
- $x_i' = x_i t^{w_i}$; for $1 \leq i \leq k$,
- $x_j' = x_j$. for $j > k$,

Local Expressions and Transforms of Foliations

Lemma (Local Expression of Transforms)

Let $\sigma : B \to X$ be a cobordant blowing-up with center $\mathcal{A}_J = \mathcal{O}_X[t^{1/w_i} x_i]_I$. Then for the standard derivations:

$$\sigma^*(\partial_{x_i}) = t_B^{w_i} \partial_{x_i'} \quad (i = 1, \dots, l)$$
$$\sigma^*(\partial_{x_i}) = \partial_{x_i'} \quad (i = l+1, \dots, n)$$
$$\sigma^c(\partial_{x_i}) = \sigma^s(\partial_{x_i}) = \partial_{x_i'} \quad (i = 1, \dots, n)$$

Transforms of Foliations: For a foliation $\mathcal{F} = \mathrm{span}_{\mathcal{O}_X}(\partial_1, \dots, \partial_r)$, its respective transforms are simply the $\mathcal{O}_B$-spans (or $K[t^{-1}]$-spans) of the transformed derivations:

$$\sigma^\bullet(\mathcal{F}) = \mathrm{span}_{\mathcal{O}_B}\{\sigma^\bullet(\partial) \mid \partial \in \mathcal{F}\} \quad \text{for } \bullet \in \{*, c, s\}$$

Transforming Derivations: Total Transform

Let $B \to X$ be a full cobordant blow-up and

$$W = [B_+/G_m] \to X$$

the associated stacky weighted blow-up of $\mathcal{J}$.
Let

$$\tau_1 : W_1 \to X$$

be the restriction to the x_1'-chart. Then locally τ_1 is given by

$$\begin{aligned} x_1 &= y^{w_1}, \\ x_2 &= x_2' y^{w_2}, \\ &\vdots \\ x_k &= x_k' y^{w_k}. \end{aligned}$$

Transforming Derivations: Total Transform

Because τ_1 is a generically finite morphism, we compute the **Total Transform** of the standard derivations via the field extension. They pick up poles along the exceptional divisor $y_1 = 0$:

$$\tau_1^*(\partial_{x_1}) = \frac{1}{w_1 y_1^{w_1}} \left(y_1 \partial_{y_1} - \sum_{i=2}^{k} w_i\, y_i \partial_{y_i} \right)$$

$$\tau_1^*(\partial_{x_i}) = \frac{1}{y_1^{w_1}} \partial_{y_i}, \quad i = 2, \ldots, k$$

$$\tau_1^*(\partial_{x_i}) = \partial_{y_i}, \quad i = k+1, \ldots, n$$

Controlled and Strict Transforms of Derivations

This yields the canonical **Controlled** and **Strict** transforms for derivations on W_1 represented stacky weighted blow-up $X = B_+/G_m$:

$$\tau_1^c(\partial_{x_1}) = \tau_1^s(\partial_{x_1}) = \frac{1}{w_1}\left(y_1\partial_{y_1} - \sum_{i=2}^{k} w_i\, y_i\partial_{y_i}\right)$$

$$\tau_1^c(\partial_{x_i}) = \tau_1^s(\partial_{x_i}) = \partial_{y_i}, \quad i = 2, \ldots, n$$

Descent of Transforms (1/2)

How the folaitions descend under group actions?

Descent under Group Actions: While pushing forward a foliation is generally not well-defined, we *can* descend a T-invariant foliation $\mathcal{F}$ under a locally free quotient $X \to Y = X/T$.

- The descent $\mathcal{F}_{|Y}$ is generated by pushing forward the T-invariant derivations.
- The kernel of this descent is precisely the relative tangent sheaf: $\mathcal{D}^{\log}_{X/Y} \cap \mathcal{F}$.

Descent of Transforms (2/2): Stack-Theoretic Blow-ups

Let $\sigma : B_+ \to Y$ be a cobordant blow-up, with its corresponding stack-theoretic weighted blow-up being the quotient $\sigma_X : X = [B_+/T] \to Y$.

Controlled Transform Descent: The descent of the controlled transform $\sigma^c(\mathcal{G})$ on B_+ precisely yields the controlled transform on the stack-theoretic blow-up $X = [B_+/T]$:

$$\sigma^c(\mathcal{G})_{|X} = \sigma^c_X(\mathcal{G})$$

Compatibility of Descent for Foliations

Lemma. Let

$$\sigma : B_+ \to X$$

be the cobordant blow-up of an $\mathcal{F}$-aligned center $\mathcal{A}_{\mathcal{J}}$, and

$$\tau : W = [B_+/G_m] \to X$$

the associated weighted blow-up.
The descent of the controlled/strict transforms agrees with the transforms on the stack:

$$(\sigma^c(\mathcal{F}))_{|W} = \tau^c(\mathcal{F}), \qquad (\sigma^s(\mathcal{F}))_{|W} = \tau^s(\mathcal{F}).$$

In particular on the chart $\tau_1 : W_1 \to X$,

$$\left(\sigma^c(\partial_{x_i})\right)_{|W_1} = \tau_1^c(\partial_{x_i}) = \partial_{x_i'} \qquad (i > 1),$$

and

$$\partial_{x_1'} = \frac{1}{w_1}\left(y\,\partial_y - \sum_{i=2}^{k} w_i\, x_i' \partial_{x_i'}\right).$$

Embedded Resolution of foliated varieties

Let $Y \subset X$ be a closed subvariety in a smooth foliated variety $(X, \mathcal{F}, E)$.

- The maximum value of the invariant drops after each blow-up.

$$\max \operatorname{inv}_{\mathcal{F}}(\sigma^c(\mathcal{I}_Y) < \max \operatorname{inv}_{\sigma^c(\mathcal{F})}(\mathcal{I}_Y).$$

- The process continues until the invariant reaches the smooth point value:

$$\operatorname{inv}_p(\mathcal{I}_Y) = (1, \ldots, 1, \infty + 1, \ldots, \infty + 1).$$

- At this stage, the singularities on the strict transform of Y are resolved, and it is $\mathcal{F}$-aligned. Thus locally in the coordinate system $x_1, \ldots, x_k, y_1, \ldots, y_{n-k}$,

$$Y = V(x_1, \ldots, x_k, y_1, \ldots, y_m) \quad m \leq n-k \quad \text{and}$$

$$\widehat{\mathcal{F}} = \widehat{\mathcal{O}}_{X,(x)} \cdot \mathcal{F} = \mathrm{span}(\partial_{x_1}, \ldots, \partial_{x_k}, \nabla_1(y), \ldots, \nabla_r(y))$$

where $V(x_1, \ldots, x_n)$ is $\mathcal{F}$- transverse and $V(y_1, \ldots, y_m)$ is $\mathcal{F}$-tangential so $\mathcal{F}(y_1, \ldots, y_m) \subseteq (y_1, \ldots, y_m)$.

Theorem ((ABTW) Embedded Resolution of foliated varieties)

Let $(X, \mathcal{F}, E)$ be a smooth foliated logarithmic variety, and let $Y \subset X$ be a subvariety. There exists a sequence of weighted (or cobordant) blow-ups:

$$(X_0, \mathcal{F}_0, E_0) \leftarrow (X_1, \mathcal{F}_1, E_1) \leftarrow \cdots \leftarrow (X_k, \mathcal{F}_k, E_k) = (X', \mathcal{F}', E'),$$

such that:

- *The strict transform Y' of Y is smooth, has normal crossings with E' is $\mathcal{F}'$-aligned and thus decomposes into $\mathcal{F}'$-transverse and $\mathcal{F}'$-tangential parts.*
- *Blow-up centers are $\mathcal{F}_i$-aligned and admissible for the strict transform of Y.*
- *If $\mathcal{F}$ is nonsingular/K-monomial, $\mathcal{F}_{i+1}$ remains such under strict/controlled transform.*
- *Additional properties from the principalization theorem apply (functoriality, compatibility with smooth morphisms, etc.).*

Definition: Transverse Section

Definition (Transverse Section)

A subvariety $Y \subset X$ is said to be *transverse* to a foliation $\mathcal{F}$ at a point $p \in Y$ if:

- There exists a partial system of parameters $(x_1, \dots, x_p)$ centered at p.
- There are derivations $\partial_{x_1}, \dots, \partial_{x_p} \in \mathcal{F} \cdot \mathcal{O}_{X,p}$
- The ideal $\mathcal{I}^Y$ of Y is locally equal to $(x_1, \dots, x_p)$.

Transverse Section: If the rank of $\mathcal{F}$ is equal to p, thus

$$\mathcal{F}_p = \mathrm{span}_{\mathcal{O}_{X,p}}(\partial_{x_1}, \dots, \partial_{x_p})$$

then $Y \subset X$ is called the *transverse section* of $\mathcal{F}$ at p.

Resolution Preserving Transverse Locus

Theorem

(ABTW)[Resolution Preserving Transverse Locus] Let $(X, \mathcal{F}, E)$ be a smooth foliated logarithmic variety. Let $Y \subset X$ be a closed subvariety generically transverse to $\mathcal{F}$.

There exists an embedded desingularization

$$(X_0, \mathcal{F}_0, E_0) \leftarrow (X_1, \mathcal{F}_1, E_1) \leftarrow \cdots \leftarrow (X_k, \mathcal{F}_k, E_k) = (X', \mathcal{F}', E'),$$

such that the strict transform Y' of Y is transverse to $\mathcal{F}'$.

- *If $\dim(Y) = \operatorname{rank}(\mathcal{F})$, then Y' is a transverse section to $\mathcal{F}'$.*
- *The sequence defines an isomorphism over the points of X where Y is $\mathcal{F}$-transverse.*

Sketch of the Proof *Y transverse to $\mathcal{F}$ at p means $\operatorname{inv}_p(\mathcal{F}) = (1, \dots 1)$*

Canonical Nonsingular Cobordant Resolution

Lemma (Resolution of $\mathcal{K}$-Monomial Foliations)

A canonical nonsingular cobordant resolution exists for a $\mathcal{K}$-monomial (and, in particular, $\mathcal{K}$-log-smooth) foliation. This resolution is compatible with smooth morphisms, group actions, and field extensions.

Sketch of the Proof & Rank Dynamics:

- **Definition:** The *smooth rank*, sm-rank$(\mathcal{F})$, at a point p is the maximal rank of a nonsingular subfoliation $\mathcal{G} \subset \mathcal{F}$ at p.
- Let $\mathcal{F}$ be generated at $p \in X$ by regular derivations ∂_{v_i} and logarithmic derivations $\nabla_j = \sum_{k=1}^{n} a_{jk} w_k \partial_{w_k}$. Let $\sigma : B_+ \to X$ be the cobordant blow-up at the maximal monomial center $V(w_i \mid w_i(p) = 0)$.
- After the cobordant blow-up, on the chart $w_k(p) \neq 0)$ derivation: $w_k \partial_{w_k}$ becomes regular, increasing smooth rank.

Resolution of Rationally Totally and $\mathcal{K}$-Darboux Integrable Foliations

Theorem

(ABTW)[Resolution of Totally Integrable Foliations] There exists a nonsingular cobordant resolution:

$$(X', \mathcal{F}', E') \to (X, \mathcal{F}, E)$$

of a Totally (resp. $\mathcal{K}$-Darboux) Integrable Foliation $(X, \mathcal{F}, E)$

- *X' admits a torus action by T, with a geometric quotient X'/T and a T-invariant projective birational morphism $\pi : X'/T \to X$.*
- *The strict transform $\mathcal{F}' := \pi^s(\mathcal{F})$ is a nonsingular T-invariant foliation on X'.*
- *Over the open subset $U = X \setminus \mathrm{Sing}(\mathcal{F})$, the foliation $\mathcal{F}$ descends to a nonsingular foliation $\mathcal{F}_{|U}$.*

Orbifold Resolution

- The process induces a $\mathbb{Q}$-monomial (resp. $\mathcal{K}$)-monomial orbifold resolution of aTotally (resp. $\mathcal{K}$-Darboux) Integrable Foliation$(X, \mathcal{F}, E)$:

  $$\sigma : (X'', \mathcal{F}'', E'') \to (X, \mathcal{F}, E),$$

  where:
  - $X'' = [X'/T]$ is the stack-theoretic quotient of X' by T.
  - $\mathcal{F}''$ is the strict transform of $\mathcal{F}$.
  - E'' is a simple normal crossing (SNC) divisor descending from E'.
- $\mathcal{F}''$ on X'' becomes a $\mathcal{K}$-monomial foliation.

Functoriality

- The resolution process is functorial for:
  - Field extensions.
  - Smooth morphisms with respect to the pair $(\mathcal{F}, \varphi)$.

Resolution of Darboux Integrable Foliations

Let $\phi : X \dashrightarrow B$ be a dominant rational map, and $\mathcal{F} = \phi^{-1}(\mathcal{G})$ be the pull-back of the $\mathcal{K}$-monomial foliation $\mathcal{G}$ on a smooth B,

- Consider the product $Y = X \times B$ with projection $p_B : Y \to B$.
- Let $\overline{X} := \Gamma(\varphi) \subset Y$ denote the graph of φ. $\overline{X} \to X$ is proper birational.

Define $\mathcal{H}$ on $Y = X \times B$ to be the inverse transform:

$$\mathcal{H} = p_B^{-1}(\mathcal{G})$$

Then $\mathcal{H}$ is $\mathcal{K}$-monomial and admits a *nonsingular cobordant resolution*.

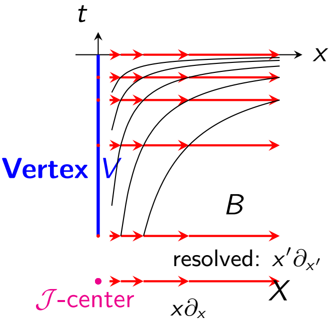
Applying Embedded Desingularization of $\overline{X} \subset Y$

- The strict transform $\widetilde{X}$ of $\overline{X} \subset Y$ becomes $\mathcal{H}$-aligned.
- The foliation $\mathcal{H}$ remains nonsingular.
- The restriction $\overline{\mathcal{F}} = \mathcal{H}_{|\overline{X}} \subset Y$is nonsingular, since $\overline{X}$ is $\mathcal{H}$-aligned.

Cobordant Resolution of a Singular Foliation

Example: The vector field $\mathcal{F} = x\partial_x$ is singular at the origin on $X = \mathbb{A}^1$.

- Consider the cobordant space $B = \mathrm{Spec}(\mathcal{O}_X[xt, t^{-1}]) = \mathrm{Spec}(\mathcal{O}_X[x', t^{-1}])$, with $x' = xt$.
- Removing the vertex $V = V(x')$ gives $B_+ = B \setminus V$.
- The strict transform becomes $\mathcal{F}' = x'\partial_{x'}$, which is nonsingular on B_+.



Cobordant blow-up removes the singularity and produces a nonsingular foliation