REVIEW OF SINGULAR COHOMOLOGY

We begin with a brief review of some basic facts about singular homology and cohomology. For details and proofs, we refer to [Mun84]. We then discuss the Leray-Hirsch theorem and the Thom isomorphism, and finally, we carry out some simple computations: the cohomology of a projective space and that of a smooth blow-up.

1. Basic facts about singular homology and cohomology

For every Abelian group A and every non-negative integer p, we have a covariant functor $H_p(-,A)$ and a contravariant functor $H^p(-,A)$ from the category of topological spaces to the category of Abelian groups. This is the p^{th} *singular homology group* (respectively, the p^{th} *singular cohomology group*) with coefficients in A. More generally, we have a covariant functor $H_p(-,-,A)$ and a contravariant functor $H^p(-,-,A)$ from the category of pairs of topological spaces[1] to the category of Abelian groups. This is the p^{th} *relative homology group* (respectively, the p^{th} *relative cohomology group*) with coefficients in A. If $f\colon X\to Y$ is a continuous map, the morphism induced in homology is denoted by f_* and that induced in cohomology is denoted by f^*.

These functors satisfy the following properties:

i) If X consists of one point, then

$$H_0(X,A)\simeq A \quad \text{and} \quad H^0(X,A)\simeq A,$$

and

$$H_p(X,A)=0=H^p(X;A) \quad \text{for} \quad p>0.$$

ii) If $X=\bigsqcup_{i\in I}X_i$, then

$$H_p(X,A)\simeq\bigoplus_{i\in I}H_p(X_i,A) \quad \text{for all} \quad p\geq 0, \text{ and}$$

$$H^p(X,A)\simeq\prod_{i\in I}H^p(X_i,A) \quad \text{for all} \quad p\geq 0.$$

iii) If $f,g\colon X\to Y$ are homotopic continuous maps, then they induce the same maps in homology $H_p(X,A)\to H_p(Y,A)$ and in cohomology $H^p(Y,A)\to H^p(X,A)$.

iv) For every pair (X,Y), we have the following functorial long exact sequences:

$$\ldots\to H_p(Y,A)\to H_p(X,A)\to H_p(X,Y,A)\to H_{p-1}(Y,A)\to\ldots$$

and

$$\ldots\to H^p(X,Y,A)\to H^p(X,A)\to H^p(Y,A)\to H^{p+1}(X,Y,A)\to\ldots.$$

[1] A pair (X,Y) consists of a topological space X and a subspace Y; a morphism in this categoy from (X,Y) to (X',Y') is given by compatible continuous maps $X\to X'$ and $Y\to Y'$.

v) (Excision) If (X, Y) is a pair and U is a subset of X such that $\overline{U}$ is contained in the interior of Y, then the inclusion induces isomorphisms

$$H_p(X \smallsetminus U, Y \smallsetminus U, A) \simeq H_p(X, Y, A) \quad \text{and}$$
$$H^p(X, Y, A) \simeq H^p(X \smallsetminus U, Y \smallsetminus U, A) \quad \text{for all} \quad p \geq 0.$$

In what follows we will mostly be interested in the case when the (co)homology has coefficients in $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, or $\mathbf{C}$. An immediate consequence of the definition of singular homology and cohomology groups is that if $\pi_0(X)$ is the set of path-components of X, then

$$H_0(X, A) \simeq A^{(\pi_0(X))} \quad \text{and} \quad H^0(X, A) \simeq \mathrm{Hom}_{\mathbf{Z}}(A, \mathbf{Z})^{\pi_0(X)}. \tag{1}$$

In particular, we have a canonical morphism of Abelian groups

$$\deg \colon H_0(X, A) \to A,$$

which is an isomorphism if X is path-connected.

A useful tool for computing cohomology is provided by the Mayer-Vietoris sequence. Suppose that U and V are open subsets of X such that $X = U \cup V$. If the following maps are given by inclusions:

$$i_U \colon U \hookrightarrow X, \; j_U \colon U \cap V \hookrightarrow U, \; i_V \colon V \hookrightarrow X, \; j_V \colon U \cap V \hookrightarrow V,$$

then for every Abelian group A, we have a long exact sequence:

$$\ldots \longrightarrow H^p(X, A) \xrightarrow{\alpha} H^p(U, A) \oplus H^p(V, A) \xrightarrow{\beta} H^p(U \cap V, A) \longrightarrow H^{p+1}(X, A) \longrightarrow \ldots,$$

where α and β are given by

$$\alpha(y) = (i_U^*(y), i_V^*(y)) \quad \text{and} \quad \beta(y_1, y_2) = j_U^*(y_1) - j_V^*(y_2).$$

A similar sequence holds for homology.

The following is the Universal Coefficient theorem, which describes the groups $H^p(X, A)$ and $H_p(X, A)$ in terms of the groups $H_p(X, \mathbf{Z})$ (see [Mun84, Theorems 55.1, 56.1].

Theorem 1.1. *For every Abelian group A and for every $p \geq 0$, we have canonical short exact sequences*[2]*:*

$$0 \to \mathrm{Ext}^1_{\mathbf{Z}}(H_{p-1}(X, \mathbf{Z}), A) \to H^p(X, A) \to \mathrm{Hom}_{\mathbf{Z}}(H_p(X, \mathbf{Z}), A) \to 0$$

and

$$0 \to H_p(X, \mathbf{Z}) \otimes_{\mathbf{Z}} A \to H_p(X, A) \to \mathrm{Tor}_1^{\mathbf{Z}}(H_{p-1}(X, \mathbf{Z}), A) \to 0.$$

Both sequences are split, but the splitting is non-canonical.

Remark 1.2. An immediate consequence of the theorem is that if K is a field containing $\mathbf{Q}$, then

$$H_p(X, K) \simeq H_p(X, \mathbf{Z}) \otimes_{\mathbf{Z}} K \quad \text{and} \quad H^p(X, K) \simeq H^p(X, \mathbf{Z}) \otimes_{\mathbf{Z}} K \simeq H_p(X, K)^*.$$

[2] We make here the convention that $H_{-1}(X, \mathbf{Z}) = 0$.

Remark 1.3. Note that if M is a finitely generated Abelian group, then $\mathrm{Ext}^1_{\mathbf{Z}}(M,\mathbf{Z})$ is torsion and $\mathrm{Hom}_{\mathbf{Z}}(M,\mathbf{Z})$ has no torsion. We thus deduce from Theorem 1.1 that if all homology groups $H_i(X,\mathbf{Z})$ are finitely generated, then the torsion subgroup $H^p(X,\mathbf{Z})_{\mathrm{tors}}$ of $H^p(X,\mathbf{Z})$ is isomorphic to $\mathrm{Ext}^1_{\mathbf{Z}}(H_{p-1}(X,\mathbf{Z}),\mathbf{Z})$, and

$$H^p(X,\mathbf{Z})/H^p(X,\mathbf{Z})_{\mathrm{tors}} \simeq \mathrm{Hom}_{\mathbf{Z}}(H_p(X,\mathbf{Z}),\mathbf{Z}).$$

If R is a ring, then

$$H^*(X,R) := \bigoplus_{p\geq 0} H^p(X,R)$$

has the structure of a skew-commutative graded ring with respect to the *cup-product*

$$H^p(X,R)\times H^q(X,R)\ni (a,b)\to a\cup b\in H^{p+q}(X,R).$$

We also have a *cap product* map

$$H^p(X,R)\times H_q(X,R)\to H_{q-p}(X,R).$$

This makes $H_*(X,R) := \bigoplus_p H_p(X,R)$ a left module over $H^*(X,R)$. These operations satisfy the *projection formula*: if $f\colon X\to Y$ is a continuous map, then

$$f_*\big(f^*(\alpha)\cap\beta\big) = \alpha\cap f_*(\beta)\quad \text{for every}\quad \alpha\in H^*(Y,R), \beta\in H_*(X,R). \tag{2}$$

One can define the cup-product more generally for relative cohomology: if R is a ring and $Y_1, Y_2\subseteq X$, then we have a cup-product map

$$H^p(X,Y_1,R)\times H^q(X,Y_2,R)\to H^{p+q}(X,Y_1\cup Y_2,R)$$

that extends the one in the absolute case and which satisfies similar properties.

Given two topological spaces X and Y, the Künneth theorem computes the (co)homology of the product $X\times Y$ in terms of the (co)homologies of X and Y. The simplest form concerns homology and says that for every $m\geq 0$, there is a natural short exact sequence

$$0\to \bigoplus_{p+q=m} H_p(X,\mathbf{Z})\otimes_{\mathbf{Z}} H_q(Y,\mathbf{Z})\to H_m(X\times Y,\mathbf{Z})\to \bigoplus_{p+q=m-1}\mathrm{Tor}_1^{\mathbf{Z}}\big(H_p(X,\mathbf{Z}),H_q(Y,\mathbf{Z})\big)\to 0,$$

which is split, but non-canonically. If all homology groups $H_p(X,\mathbf{Z})$ are finitely generated, then for every $m\geq 0$ we have a natural short exact sequence for cohomology:

$$0\to \bigoplus_{p+q=m} H^p(X,\mathbf{Z})\otimes_{\mathbf{Z}} H^q(Y,\mathbf{Z})\xrightarrow{\vartheta} H^m(X\times Y,\mathbf{Z})\to \bigoplus_{p+q=m+1}\mathrm{Tor}_1^{\mathbf{Z}}\big(H^p(X,\mathbf{Z}),H^q(Y,\mathbf{Z})\big)\to 0,$$

where $\vartheta(\alpha\otimes\beta)=\mathrm{pr}_1^*(\alpha)\cup\mathrm{pr}_2^*(\beta)$. Moreover, this sequence splits (non-canonically) if also all $H^q(Y,\mathbf{Z})$ are finitely generated. For these results, see [Mun84, §59, 60].

Suppose now that M is a compact, orientable, n-dimensional smooth manifold. The orientation on M defines on M a *fundamental class* $\mu_M\in H_n(M,\mathbf{Z})$. The following result is one of the incarnations of *Poincaré duality*.

Theorem 1.4. *With the above notation, the cap product map*

$$H^p(M,\mathbf{Z})\to H_{n-p}(M,\mathbf{Z}),\quad \alpha\to\alpha\cap\mu_M \tag{3}$$

is an isomorphism.

One can show that for a compact manifold M, the groups $H_p(M,\mathbf{Z})$ (hence also the groups $H^p(M,\mathbf{Z})$) are finitely generated. We can now rephrase Poincaré duality as follows. Consider the following bilinear pairing

$$H^p(X,\mathbf{Z})\times H^{n-p}(X,\mathbf{Z})\to\mathbf{Z},\quad (\alpha,\beta)\to \deg\big((\alpha\cup\beta)\cap\mu_M\big). \tag{4}$$

Via the isomorphism (3), this gets identified with the morphism

$$H^p(X,\mathbf{Z})\times H_p(X,\mathbf{Z})\to\mathbf{Z},\quad (\alpha,\beta)\to\deg(\alpha\cap\beta).$$

Using the Universal Coefficient theorem, we see that the map induced after killing the torsion is a perfect pairing. We thus conclude that (4) induces after killing the torsion a perfect pairing:

$$H^p(X,\mathbf{Z})/H^p(X,\mathbf{Z})_{\rm tors}\times H^{n-p}(X,\mathbf{Z})/H^{n-p}(X,\mathbf{Z})_{\rm tors}\to\mathbf{Z}. \tag{5}$$

By tensoring with $\mathbf{Q}$, we obtain a perfect pairing

$$H^p(X,\mathbf{Q})\times H^{n-p}(X,\mathbf{Q})\to\mathbf{Q},\quad (\alpha,\beta)\to\deg\big((\alpha\cup\beta)\cap\mu_M\big).$$

2. The Leray-Hirsch theorem and the Thom isomorphism

We discuss two results that we will need later, in order to compute the cohomology of projective bundles and that of smooth blow-ups. We begin with a result that allows the description of the cohomology for the total space of a locally trivial map. Recall that a continuous map $f\colon Y\to X$ is *locally trivial, with fiber* F, if X can be covered by open subsets U such that $f^{-1}(U)$ is homeomorphic over U with $U\times F$. In the cases of interest to us, X will be covered by finitely many such subsets, and in this case the results below are easier to prove.

Theorem 2.1. *Let $f\colon Y\to X$ be a continuous map, which is locally trivial, with fiber F. Suppose that all cohomology groups $H^p(F,\mathbf{Z})$ are finitely generated, free Abelian groups. If $\alpha_1,\ldots,\alpha_r\in H^*(Y,\mathbf{Z})$ are such that for every $x\in X$, the inclusion $i_x\colon f^{-1}(x)\hookrightarrow Y$ has the property that $i_x^*(\alpha_1),\ldots,i_x^*(\alpha_r)$ give a basis of $H^*(f^{-1}(x),\mathbf{Z})$, then we have an isomorphism of Abelian groups*

$$\mathbf{Z}^r\otimes_{\mathbf{Z}}H^*(X,\mathbf{Z})\to H^*(Y,\mathbf{Z}),\ \ (m_1,\ldots,m_r)\otimes\beta\to\left(\sum_{i=1}^r m_i\alpha_i\right)\cup f^*(\beta).$$

Proof. If $Y=X\times F$, then the assertion is an easy consequence of the Künneth theorem. When X has a finite cover by open subsets U_i such that $f^{-1}(U_i)$ is isomorphic over U_i with $U_i\times F$, one argues by induction on the number of open subsets, using the Mayer-Vietoris sequence. We do not give the proof in the general case, as we will not need it. □

We now turn to the definition of the Thom class and to the statement of the Thom isomorphism theorem. Suppose that $\pi\colon E\to X$ is an oriented, real vector bundle of rank $r\geq 1$, on a topological space X. In particular, for every $x\in X$, the fiber $E_x=\pi^{-1}(x)$ is an r-dimensional, oriented real vector space. We will consider X embedded in E via the 0-section.

Recall that

$$H^r(\mathbf{R}^r, \mathbf{R}^r \smallsetminus \{0\}, \mathbf{Z}) \simeq \mathbf{Z},$$

and choosing a generator is equivalent to the choice of an orientation on $\mathbf{R}^r$. Indeed, it follows from the long exact sequence in cohomology for $(\mathbf{R}^r, \mathbf{R}^r \smallsetminus \{0\})$ and from the fact that $\mathbf{R}^r$ is contractible, while $\mathbf{R}^r \smallsetminus \{0\}$ is homotopically equivalent to the sphere S^{r-1}, that[3]

$$H^r(\mathbf{R}^r, \mathbf{R}^r \smallsetminus \{0\}, \mathbf{Z}) \simeq H^{r-1}(\mathbf{R}^r \smallsetminus \{0\}, \mathbf{Z}) \simeq H^{r-1}(S^{r-1}, \mathbf{Z}) \simeq \mathbf{Z}.$$

Theorem 2.2. *Let $\pi\colon E \to X$ be an oriented, real vector bundle of rank $r \geq 1$, on a topological space X.*

i) *There is a unique $\eta_E \in H^r(E, E \smallsetminus X, \mathbf{Z})$ (the* Thom class *of E) such that for every $x \in X$, the restriction of η_E to $H^r(\pi^{-1}(x), \pi^{-1}(x) \smallsetminus \{0\}, \mathbf{Z})$ corresponds to the given orientation on the fiber.*

ii) *For every closed subset W of X and every $p \geq 0$, we have*

$$H^p(E, E \smallsetminus W, \mathbf{Z}) = 0 \quad \text{for} \quad p < r$$

and the map

$$H^{p-r}(X, X \smallsetminus W, \mathbf{Z}) \to H^p(E, E \smallsetminus W, \mathbf{Z}), \quad \alpha \to \pi^*(\alpha) \cup \eta_E$$

is an isomorphism for every $p \geq r$ (the Thom isomorphism*).*

Proof. We only sketch the argument under the assumption that there are finitely many open subsets of X that cover X and such that E is trivial over each of them (this is enough for our purpose). For both i) and ii), arguing by induction on the number of open subsets and using the Mayer-Vietoris sequence, we reduce to the case when E is trivial. In this case the assertions follow easily from the Künneth theorem. □

Remark 2.3. We will apply the Thom isomorphism via the following consequence. Suppose that X is an oriented, n-dimensional smooth manifold, and Y is a closed, oriented submanifold, of codimension r. The orientations on the tangent bundles T_X and T_Y induce an orientation on the normal bundle $N = N_{Y/X}$, since we have a canonical isomorphism $\det(T_X) \simeq \det(T_Y) \otimes \det(N)$. Recall that by the tubular neighborhood theorem, there is an open neighborhood U of Y in X, a retract $r\colon U \to Y$, and a homeomorphism $h\colon U \to N$, such that we have a commutative diagram

$$\begin{array}{ccccc} Y & \xrightarrow{j} & U & \xrightarrow{r} & Y \\ \downarrow{\scriptstyle 1_Y} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle 1_Y} \\ Y & \xrightarrow{i} & N & \xrightarrow{\pi} & Y, \end{array}$$

where j is the inclusion, i is the embedding as the 0-section, and π is the vector bundle map. If W a a closed subset of Y, we then obtain isomorphisms

$$H^{p-r}(Y, Y \smallsetminus W, \mathbf{Z}) \simeq H^p(N, N \smallsetminus W, \mathbf{Z}) \simeq H^p(U, U \smallsetminus W, \mathbf{Z}) \simeq H^p(X, X \smallsetminus W, \mathbf{Z}),$$

[3] In order for this to also hold for $r = 1$, we need to replace the 0^{th} cohomology by the reduced version, the cokernel of the canonical map $\mathbf{Z} \to H^0(-, \mathbf{Z})$.

for every $p \geq 0$, where the first isomorphism is provided by Theorem 2.2, the second one is induced by h, and the third one is given by excision (we make the convention that the first group is 0 for $p < r$). In particular, we have isomorphisms

$$H^{p-r}(Y, \mathbf{Z}) \simeq H^p(X, X \smallsetminus Y, \mathbf{Z}) \quad \text{for all} \quad p \geq 0. \tag{6}$$

Remark 2.4. If X and Y are as in the previous remark and $\iota\colon Y \to X$ is the inclusion, then the Gysin map

$$\iota_*\colon H^p(Y, \mathbf{Z}) \to H^{p+r}(X, \mathbf{Z})$$

is defined as the composition

$$H^p(Y, \mathbf{Z}) \to H^{p+r}(X, X \smallsetminus Y, \mathbf{Z}) \to H^{p+r}(X, \mathbf{Z}),$$

where the first map is the isomorphism in (6) and the second one is the canonical map induced by $(X, \emptyset) \to (X, X \smallsetminus Y)$. Therefore the long exact sequence in cohomology for the pair $(X, X \smallsetminus Y)$ becomes

$$\ldots \to H^{p-r}(Y, \mathbf{Z}) \xrightarrow{\iota_*} H^p(X, \mathbf{Z}) \to H^p(X \smallsetminus Y, \mathbf{Z}) \to H^{p-r+1}(Y, \mathbf{Z}) \to \ldots. \tag{7}$$

One can show that Gysin maps are functorial for closed embeddings of oriented smooth manifolds.

Remark 2.5. If $f\colon Y \to X$ is any smooth map between compact, oriented smooth manifolds, with $\dim(X) = n$ and $\dim(Y) = m$, we get a morphism

$$f_*\colon H^p(Y, \mathbf{Z}) \to H^{p+r}(X, \mathbf{Z}),$$

where $r = n - m$. Indeed, Poincaré duality gives isomorphisms

$$H^p(Y, \mathbf{Z}) \simeq H_{m-p}(Y, \mathbf{Z}) \quad \text{and} \quad H^{p+r}(X, \mathbf{Z}) \simeq H_{m-p}(X, \mathbf{Z})$$

and via these isomorphisms, f_* corresponds to the morphism in homology associated to f. It is clear that this is functorial. Moreover, one can show that if f embeds Y as a submanifold of X, then this definition coincides with the one in Remark 2.4.

3. A few facts about the cohomology of complex algebraic varieties

If Y is a complete complex algebraic variety of dimension n, then Y^{an} admits a finite triangulation, with simplices of dimension $\leq 2n$. More generally, if Z is a closed subvariety of Y, then Z^{an} and Y^{an} admit compatible finite triangulations, with simplices of dimension $\leq 2n$. Given an arbitrary complex algebraic variety X, of dimension n, it can be embedded as an open subvariety of a complete variety Y by Nagata's theorem. Using the above-mentioned fact about triangulations, one can show that the groups $H_i(X, \mathbf{Z})$ and $H^i(X, \mathbf{Z})$ are finitely generated, and they vanish for $i > 2n$ (note that we write $H^i(X, \mathbf{Z})$ for $H^i(X^{\mathrm{an}}, \mathbf{Z})$). The *Betti numbers* of X are the ranks of these (co)homology groups:

$$b_i(X) := \operatorname{rk} H^i(X, \mathbf{Z}) = \operatorname{rk} H_i(X, \mathbf{Z}).$$

If X is any complete complex algebraic variety of dimension n, then X carries a *fundamental class* $\mu_X \in H_{2n}(X, \mathbf{Z})$, defined as follows. If X is smooth, then X^{an} is a compact oriented smooth manifold of dimension $2n$ and μ_X is the corresponding fundamental class. Suppose now that X is an arbitrary complete complex n-dimensional variety. By

Hironaka's theorem, we have a proper birational morphism $f\colon Y \to X$ such that Y is smooth, and we put $\mu_X = f_*(\mu_Y) \in H_{2n}(X, \mathbf{Z})$.

Lemma 3.1. *The definition of μ_X does not depend on the choice of resolution.*

Sketch of proof. Standard arguments reduce the statement to the following: if $\pi\colon Z \to Y$ is a birational morphism of smooth, complete, n-dimensional complex algebraic varieties, then $\pi_*\mu_Z = \mu_Y$. Since $H_{2n}(Y, \mathbf{Z}) \simeq \mathbf{Z}$, it follows that we have $\pi_*\mu_Z = d\mu_X$ for some integer d. In order to show that $d = 1$, it is enough to pass to cohomology with $\mathbf{R}$-coefficients. Note that we have an isomorphism

$$H_{2n}(Y, \mathbf{R}) \simeq H^{2n}_{\mathrm{DR}}(Y)^*$$

such that μ_Y corresponds to the map $[\omega] \to \int_Y \omega$ and a similar assertion holds for Z. Since we have

$$\int_Y \omega = \int_Z \pi^*(\omega)$$

(we use here that π is almost everywhere a diffeomorphism), we conclude that $d = 1$. □

If $g\colon W \to X$ is an arbitrary surjective, generically finite morphism of complete complex algebraic varieties, then a similar argument to that used in the proof of the above lemma gives

$$f_*(\mu_W) = \deg(f)\mu_X. \tag{8}$$

On the other hand, if g is such that $\dim\big(g(W)\big) < \dim(W)$, then $g_*(\mu_W) = 0$ (this is due to the fact that g_* factors through $H_*\big(g(W), \mathbf{Z}\big)$ and $H_i\big(g(W), \mathbf{Z}\big) = 0$ for $i > 2\cdot\dim\big(g(W)\big)$.

Suppose now that X is a smooth, complete complex algebraic variety and W is a codimension r irreducible subvariety We define the *cohomology class* $[W] \in H^{2r}(X, \mathbf{Z})$ as follows. If $n = \dim(X)$ and $i\colon W \hookrightarrow X$ is the inclusion, then $[W]$ is the element corresponding to $i_*\mu_W \in H_{2n-2r}(X, \mathbf{Z})$ via the Poincaré duality isomorphism $H^{2r}(X, \mathbf{Z}) \simeq H_{2n-2r}(X, \mathbf{Z})$.

Recall that if $f\colon W \to X$ is a morphism of smooth, complete complex algebraic varieties, with $\dim(W) = m$ and $\dim(X) = n$, then for every p we have a morphism

$$f_*\colon H^p(W, \mathbf{Z}) \to H^{p+2r}(X, \mathbf{Z}),$$

where $r = m - n$. This is induced via Poincaré duality by the push-forward in homology (see Remark 2.5). Using the behavior of the Hodge decomposition with respect to Poincaré duality, it is straightforward to check that after tensoring with $\mathbf{C}$, f^* maps $H^{i,j}(W)$ to $H^{i+r,j+r}(X)$.

It is easy to deduce from (8) and the projection formula (2) that if f is generically finite and surjective (hence $m = n$), then $f_* \circ f^* = \deg(f)\cdot\mathrm{Id}$. In particular, f^* is injective if f is birational, and it is injective with $\mathbf{Q}$-coefficients if f is generically finite and surjective.

In fact, a more general result holds: if $f\colon W \to X$ is a surjective morphism between complete, smooth, complex algebraic varieties, then

$$f^*\colon H^p(X, \mathbf{Q}) \to H^p(W, \mathbf{Q})$$

is injective for all p. Since every complete, smooth, complex algebraic variety admits a birational morphism from a projective, smooth variety, which is Kähler, the assertion follows from the following result. While we do not need it in what follows, we give a proof following [Voi07], since it illustrates some concepts we discussed in class.

Proposition 3.2. *If $f\colon Y \to X$ is a holomorphic map between compact, complex manifolds, with Y being Kähler, then*

$$f^*\colon H^p(X, \mathbf{Q}) \to H^p(Y, \mathbf{Q})$$

is injective for all p

Proof. Recall that we say that a real (n,n)-form η on an n-dimensional complex manifold Z is positive if on every chart with coordinates $z_1, \ldots, z_n$, if we write $\eta = hi^n dz_1 \wedge d\overline{z_1} \wedge \ldots \wedge dz_n \wedge d\overline{z_n}$ (so that h is a real smooth function), then $h(p) > 0$ for every point p in the chart. Note that if Z is compact and η is an (n,n)-form on Z that is positive on an open subset of Z whose complement has measure 0, then we clearly have $\int_Z \eta > 0$.

A similar definition applies, more generally, if E is a rank r vector bundle on Z and η is a section of $\wedge^r E^*_{\mathbf{R}}$. We have seen in class that if h is a hermitian metric on E and ω_h is the corresponding real $(1,1)$-form, then ω_h^r is positive (in fact, it is a positive multiple of the volume element associated to h).

Of course, it is enough to prove the assertion in the proposition with real coefficients; in this case, we identify the singular cohomology to the corresponding De Rham cohomology. Let $n = \dim(X)$ and $m = \dim(Y)$. We put $r = m - n$. The key point is to show the assertion in the proposition for $p = 2n$. In order to do this, we choose a positive (n,n)-form η on X (for example, constructed via a hermitian metric on T_X, as above). Since $\int_X \eta \neq 0$, it follows that $[\eta]$ is a generator for $H^{2n}_{\rm DR}(X)$ and we need to show that $[f^*\eta] \in H^{2n}_{\rm DR}(Y)$ is nonzero. Let ω be a Kähler form on Y. If $f^*(\eta)$ is exact, then $f^*(\eta) \wedge \omega^r$ is exact (here is where we use that ω is closed). Hence it is enough to show that $\int_Y f^*(\eta) \wedge \omega^r \neq 0$.

Note that we have an open subset U of Y such that f is submersion on U and such that $Y \smallsetminus U$ has measure 0 (by Sard's theorem). As we have seen, in order to conclude the proof of the case $p = 2n$, it is enough to show that $f^*(\eta) \wedge \omega^r$ is positive on U. Since f is a submersion on U, after restricting to a suitable neighborhood of a given point in U, we may assume that $Y = W \times X$, for a complex manifold W, such that f is the projection onto the second component. If g is the other projection, then we have a decomposition $T^*_Y \simeq g^*(T^*_W) \oplus f^*(T^*_X)$. Let ω_1 be the projection of ω onto $\wedge^2 g^*(T^*W)$. Note that we have

$$f^*(\eta) \wedge \omega^r = f^*(\eta) \wedge \omega_1^r.$$

Moreover, the hermitian metric on T_X associated to ω restricts to a hermitian metric on $g^*(T_W)$, whose associated $(1,1)$-form is ω_1. This implies that ω_1^r is positive, as a section of $\wedge^r g^*(T^*_W)$. Since $f^*(\eta)$ is also positive, as a section of $\wedge^r f^*(T^*_X)$, it is clear that $f^*(\eta) \wedge \omega_1^r$ is positive. This completes the assertion in the case $p = 2n$.

The general case follows using Poincaré duality: given any $\alpha \in H^p_{\rm DR}(X)$, it follows from Poincaré duality that there is $\beta \in H^{2n-p}_{\rm DR}(X)$ such that $\alpha \cup \beta \neq 0$. We deduce from the above special case that $f^*(\alpha) \cup f^*(\beta) = f^*(\alpha \cup \beta) \neq 0$, hence $f^*(\alpha) \neq 0$. This completes the proof of the proposition.

□

Remark 3.3. If X is a smooth, n-dimensional, complete complex variety, Y is a smooth, Cartier divisor on X, and $i\colon Y \hookrightarrow X$ is the inclusion, then one can show that

$$i_*(\mu_Y) = c^1\big(\mathcal{O}_X(Y)\big) \cap \mu_X \text{ in } H_{2n-2}(X, \mathbf{Z}). \tag{9}$$

Equivalently, in cohomology we have $[Y] = c_1\big(\mathcal{O}_X(Y)\big)$. In fact, this equality holds for every prime divisor Y in X.

Via the projection formula, we deduce from (9) that for every $\alpha \in H^p(X, \mathbf{Z})$, we have

$$i_*\big(i^*(\alpha)\big) = \alpha \cup c^1\big(\mathcal{O}_X(Y)\big) \in H^{p+2}(X, \mathbf{Z}).$$

More generally, suppose that $Y_1, \ldots, Y_r$ are smooth, effective divisors on X that intersect transversely, so that $Y = Y_1 \cap \ldots \cap Y_r$ is smooth, of codimension r in X. An easy induction on r implies that

$$[Y] = c^1\big(\mathcal{O}_X(Y_1)\big) \cup \ldots \cup c^1\big(\mathcal{O}_X(Y_r)\big).$$

Note that if $r = n$, then via the isomorphism $H^{2n}(X, \mathbf{Z}) \simeq \mathbf{Z}$, the cohomology class $[Y]$ of Y is mapped to $\#Y$. In particular, if Y is nonempty, we see that $c^1\big(\mathcal{O}_X(Y_1)\big) \cup \ldots \cup c^1\big(\mathcal{O}_X(Y_r)\big) \neq 0$.

Example 3.4. For example, if $X = \mathbf{P}^n$, then for every $d \leq n$, we have $c^1\big(\mathcal{O}_{\mathbf{P}^n}(1)\big)^d = [L]$, where $L \subseteq \mathbf{P}^n$ is any linear subspace of codimension d. In particular, via the isomorphism $H^{2n}(\mathbf{P}^n, \mathbf{Z}) \simeq \mathbf{Z}$, we see that $c^1\big(\mathcal{O}_{\mathbf{P}^n}(1)\big)^n = 1$.

4. Cohomology of projective bundles and smooth blow-ups

We begin with the computation of the cohomology of the complex projective space.

Proposition 4.1. *For every $n \geq 0$, we have a ring isomorphism*

$$H^*(\mathbf{P}^n, \mathbf{Z}) \simeq \mathbf{Z}[x]/(x^{n+1}),$$

mapping $c^1\big(\mathcal{O}_{\mathbf{P}^n}(1)\big)$ to the class of x. In particular, we have $H^i(\mathbf{P}^n, \mathbf{Z}) = 0$ if i is odd or $i > 2n$, and $H^i(\mathbf{P}^n, \mathbf{Z}) \simeq \mathbf{Z}$, otherwise.

Proof. We proceed by induction on n, following the argument in [Voi07]. The assertion is trivial for $n = 0$. In order to prove the induction step, note that we have a closed subvariety Y of $\mathbf{P}^n$ such that $Y \simeq \mathbf{P}^{n-1}$ and $\mathbf{P}^n \smallsetminus Y \simeq \mathbf{C}^n$. The long exact sequence (7) becomes

$$\ldots \to H^{i-2}(\mathbf{P}^{n-1}, \mathbf{Z}) \to H^i(\mathbf{P}^n, \mathbf{Z}) \to H^i(\mathbf{C}^n, \mathbf{Z}) \to H^{i-1}(\mathbf{P}^{n-1}, \mathbf{Z}) \to \ldots.$$

Since $\mathbf{C}^n$ is contractible, we have $H^i(\mathbf{C}^n, \mathbf{Z}) = 0$ for all $i > 0$, and we conclude from the exact sequence that

$$H^i(\mathbf{P}^n, \mathbf{Z}) \simeq H^{i-2}(\mathbf{P}^{n-1}, \mathbf{Z}) \quad \text{for all} \quad i \geq 1$$

and $H^0(\mathbf{P}^n, \mathbf{Z}) \simeq H^0(\mathbf{C}^n, \mathbf{Z}) \simeq \mathbf{Z}$. The last assertion in the proposition thus follows from the inductive assumption.

In order to complete the proof of the induction step, we need to show that if $h = c^1\big(\mathcal{O}_{\mathbf{P}^n}(1)\big)$, then h^k is a generator of $H^{2k}(\mathbf{P}^n, \mathbf{Z})$ for $0 \leq k \leq n$. The key point is that $(h^n) = 1$ (see Example 3.4). In particular, we have $h^k \neq 0$ for $0 \leq k \leq n$. For every element $\alpha \in H^{2k}(\mathbf{P}^n, \mathbf{Z})$, we can thus write $\alpha = qh^k$, for some $q \in \mathbf{Q}$. By multiplying with h^{n-k} and using the isomorphism $H^{2n}(\mathbf{P}^n, \mathbf{Q}) \simeq \mathbf{Q}$ that maps h^n to 1, we see that $q \in \mathbf{Z}$. This implies that $H^{2k}(\mathbf{P}^n, \mathbf{Z})$ is generated by α^k, completing the proof. □

Remark 4.2. Since $c_1\big(\mathcal{O}_{\mathbf{P}^n}(1)\big)$ is a cohomology class of type $(1,1)$, it follows from the description of $H^*(\mathbf{P}^n, \mathbf{Z})$ in Proposition 4.1 that $H^{p,p}(\mathbf{P}^n) = H^{2p}(\mathbf{P}^n, \mathbf{C}) \simeq \mathbf{C}$ if $0 \leq p \leq n$, and all other $H^{p,q}(\mathbf{P}^n, \mathbf{C})$ are zero.

Corollary 4.3. *If X is a complex variety and $\pi \colon \mathbf{P}(E) \to X$ is a projective bundle on X, with $\mathrm{rk}(E) = n+1$, then we have a group isomorphism*

$$H^*(X, \mathbf{Z})[x]/(x^{n+1}) \to H^*\big(\mathbf{P}(E), \mathbf{Z}\big), \ \sum_{i=0}^{n} \alpha_i x^i \to \sum_{i=0}^{n} \pi^*(\alpha_i) \cup c^1\big(\mathcal{O}_{\mathbf{P}(E)}(1)\big)^i.$$

Proof. It is clear that the map is a group homomorphism. In order to see that it is an isomorphism, it is enough to apply the Leray-Hirsch theorem for the elements $c^1\big(\mathcal{O}_{\mathbf{P}^n}(1)\big)^i \in H^{2i}(\mathbf{P}(E), \mathbf{Z})$, for $0 \leq i \leq n$; the fact that they satisfy the hypothesis in the theorem follows from Proposition 4.1. □

Remark 4.4. Since $c_1\big(\mathcal{O}_{\mathbf{P}^n}(1)\big)$ is a cohomology class of type $(1,1)$, it follows from the above corollary that for every p and q, we have

$$H^{p,q}\big(\mathbf{P}(E)\big) \simeq \bigoplus_{0 \leq i \leq n} H^{p-i,q-i}(X).$$

Exercise 4.5. With the notation in Corollary 4.3, show that π_* vanishes on $H^*(X, \mathbf{Z})x^i$, for $0 \leq i \leq n-1$ and $\pi_*(\alpha x^n) = \alpha$ for every $\alpha \in H^*(X, \mathbf{Z})$.

We can now compute the cohomology of a smooth blow-up. Let us first fix some notation. We consider a smooth, complete, complex variety X and a smooth closed subvariety Y of X, of codimension r. Let $f \colon \widetilde{X} \to X$ be the blow-up of X along Y, with exceptional divisor E. Let $j \colon E \hookrightarrow \widetilde{X}$ be the inclusion and $g \colon E \to Y$ the map induced by f.

Proposition 4.6. *With the above notation, for every $i \geq 0$, we have an isomorphism*

$$H^p(X, \mathbf{Z}) \oplus \bigoplus_{q=1}^{r-1} H^{p-2q}(Y, \mathbf{Z}) \simeq H^p(\widetilde{X}, \mathbf{Z}) \quad \textit{given by}$$

$$(\beta, \alpha_1, \ldots, \alpha_{r-1}) \to f^*(\beta) + \sum_{q=1}^{r-1} j_*\big(c^1(\mathcal{O}_E(1))^q \cup g^*(\alpha_q)\big),$$

with the convention that $H^{p-2q}(Y, \mathbf{Z}) = 0$ if $p - 2q < 0$.

Proof. Since E is a projective bundle over Y, with fibers isomorphic to $\mathbf{P}^{r-1}$, it follows from Corollary 4.3 that we have an isomorphism

$$\bigoplus_{q=1}^{r} H^{p-2q}(Y,\mathbf{Z}) \to H^{p-2}(E,\mathbf{Z}),\ \ (\alpha_1,\ldots,\alpha_r) \to \sum_{q=1}^{r} c^1\big(\mathcal{O}_E(1)\big)^{q-1} \cup g^*(\alpha_i). \tag{10}$$

By considering the long exact cohomology sequences for the two pairs $(X, X \smallsetminus Y)$ and $(\widetilde{X}, \widetilde{X} \smallsetminus E)$, we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
H^p(X, X\smallsetminus Y, \mathbf{Z}) & \longrightarrow & H^p(X,\mathbf{Z}) & \longrightarrow & H^p(X\smallsetminus Y,\mathbf{Z}) & \longrightarrow & H^{p+1}(X, X\smallsetminus Y,\mathbf{Z}) \\
\downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow & & \downarrow{\scriptstyle f^*} \\
H^p(\widetilde{X}, \widetilde{X}\smallsetminus E, \mathbf{Z}) & \longrightarrow & H^p(\widetilde{X},\mathbf{Z}) & \longrightarrow & H^p(\widetilde{X}\smallsetminus E,\mathbf{Z}) & \longrightarrow & H^{p+1}(\widetilde{X}, \widetilde{X}\smallsetminus E,\mathbf{Z}),
\end{array}$$

in which the vertical maps are induced by the pull-back in cohomology corresponding to f. Note that the third vertical map is an isomorphism. Moreover, by using Thom isomorphisms as in Remark 2.4, we obtain a commutative diagram

$$\begin{array}{ccc}
H^p(X, X\smallsetminus Y, \mathbf{Z}) & \longrightarrow & H^{p-2r}(Y,\mathbf{Z}) \\
\downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle \varphi_p} \\
H^p(\widetilde{X}, \widetilde{X}\smallsetminus E, \mathbf{Z}) & \longrightarrow & H^{p-2}(E,\mathbf{Z}),
\end{array}$$

in which the horizontal maps are isomorphisms. Moreover, one can check that $g_* \circ \varphi_p$ is the identity on $H^{p-2r}(Y,\mathbf{Z})$; in other words, via the isomorphism in (10), the last component of $\varphi_p(\alpha)$ is $c_1\big(\mathcal{O}_E(1)\big)^{r-1} \cup \alpha$ (we use here the assertion in Exercise 4.5). Using also the fact that $f^* \colon H^p(X,\mathbf{Z}) \to H^p(\widetilde{X},\mathbf{Z})$ is injective, the assertion in the proposition follows from the above exact sequences, using a straightforward diagram chase. □

References

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