# SINGULAR COHOMOLOGY AS SHEAF COHOMOLOGY WITH CONSTANT COEFFICIENTS

Given a topological space $X$ and an Abelian group $A$, we temporarily denote by $H^i_{\mathrm{sing}}(X,A)$ the $i^{\mathrm{th}}$ singular cohomology group of $X$ with coefficients in $A$. If $R$ is a commutative ring and $A$ is an $R$-module, then $H^i_{\mathrm{sing}}(X,A)$ has a natural structure of $R$-module.

Our goal is to prove is to prove the following result relating sheaf cohomology and singular cohomology on "nice" topological spaces.

**Theorem 0.1.** *If $X$ is a paracompact, locally contractible*[1] *topological space, then for every commutative ring $R$ and every $R$-module $A$, we have a canonical isomorphism of $R$-modules*

$$H^i(X,A) \simeq H^i_{\mathrm{sing}}(X,A).$$

**Remark 0.2.** Note that one can't hope to have an isomorphism as in the above theorem for all $X$. For example, we have $H^0(X,\mathbf{Z}) \simeq \mathbf{Z}^{(I_X)}$, where $I_X$ is the set of connected components of $X$, while $H^0_{\mathrm{sing}}(X,\mathbf{Z}) \simeq \mathbf{Z}^{(J_X)}$, where $J_X$ is the set of path-wise connected components of $X$.

**Remark 0.3.** An obvious example of a locally contractible space is a topological manifold. Other examples are provided by CW-complexes (see [Hat02, Proposition A.4]).

The key ingredient in the proof of the above theorem is the following general proposition about certain presheaves on paracompact spaces.

**Proposition 0.4.** *Let $X$ be a paracompact topological space and $\mathcal{F}$ a presheaf of Abelian groups on $X$ that satisfies the following condition: for every open cover $X=\bigcup_{i\in I} U_i$ and for every $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all $i$ and $j$, there is $s\in\mathcal{F}(X)$ such that $s|_{U_i} = s_i$ for all $i$. If $\mathcal{F}\to\mathcal{F}^+$ is the canonical morphism to the associated sheaf, then the morphism $\mathcal{F}(X)\to\mathcal{F}^+(X)$ is surjective.*

*Proof.* A section $s\in\mathcal{F}^+(X)$ is given by a map $s\colon X\to\sqcup_{x\in X}\mathcal{F}_x$ such that we have an open cover $X=\bigcup_{i\in I}U_i$ and sections $s_i\in\mathcal{F}(U_i)$ such that $s(x)=(s_i)_x$ for every $x\in U_i$. After passing to a refinement, we may assume that the cover is locally finite. We choose another open cover $X=\bigcup_{i\in I}U'_i$ with $\overline{U'_i}\subseteq U_i$ for all $i$. Note that if $x\in U_i\cap U_j$, then $(s_i)_x=(s_j)_x$, hence there is an open neighborhood $V_{i,j}(x)\subseteq U_i\cap U_j$ such that $s_i|_{V_{i,j}(x)} = s_j|_{V_{i,j}(x)}$.

Given any $x\in X$, we choose an open neighborhood $V(x)$ of $x$, such that the following conditions are satisfied:

1) If $x\in U_i\cap U_j$, then $V(x)\subseteq V_{i,j}(x)$.
2) If $x\in U_i$, then $V(x)\subseteq U_i$.

[1] A topological space is *locally contractible* if every point has a basis of contractible open neighborhoods.

3) If $x \in U_i'$, then $V(x) \subseteq U_i'$.
4) If $V(x) \cap \overline{U_i'} \neq \emptyset$, then $x \in \overline{U_i'}$.

This is possible since the cover given by the $U_i$ is locally finite, hence every $x$ lies in only finitely many $U_i$. Note that in this case we also have: if $x, y \in X$ are such that $V(x) \cap V(y) \neq \emptyset$, then there is $i$ such that $V(x), V(y) \subseteq U_i$. Indeed, if $x \in U_i'$, then by 3) we have $V(x) \subseteq U_i'$; therefore $V(y) \cap \overline{U_i'} \neq \emptyset$, and thus $y \in \overline{U_i'}$ by 4). We thus get $V(y) \subseteq U_i$ by 2).

For every $x \in X$, it follows from 2) that if $x \in U_i$, then $V(x) \subseteq U_i$, and we put $\alpha^{(x)} = s_i|_{V(x)}$; this does not depend on $i$ by 1). Moreover, we have seen that if $V(x) \cap V(y) \neq \emptyset$, then there is $i$ such that $V(x), V(y) \subseteq U_i$, in which case it is clear that

$$\alpha^{(x)}|_{V(x)\cap V(y)} = s_i|_{V(x)\cap V(y)} = \alpha^{(y)}|_{V(x)\cap V(y)}.$$

By hypothesis, we can find $t \in \mathcal{F}(X)$ such that $t|_{V(x)} = \alpha^{(x)}$ for all $x \in X$. In particular, we have $t_x = \alpha_x^{(x)} = s(x)$ for every $x \in X$, and thus $s = \varphi(t)$. □

We can now relate sheaf cohomology and singular cohomology.

*Proof of Theorem 0.1.* Recall that for every $p \geq 0$, a $p$-simplex in $X$ is a continuous map $\Delta^p \to X$ from the standard $p$-dimensional simplex to $X$. The group of $p$-chains in $X$, denoted $\mathcal{C}_p(X)$, is the free Abelian group on the set of $p$-simplices and the $R$-module of $p$-cochains with values in $A$, denoted $\mathcal{C}^p(X, A)$, is equal to $\mathrm{Hom}_{\mathbf{Z}}\big(\mathcal{C}_p(X), A\big)$. Therefore a $p$-cochain can be identified to a map from the set of $p$-simplices in $X$ to $A$. For every $p \geq 0$ we have maps $\partial\colon \mathcal{C}^p(X, A) \to \mathcal{C}^{p+1}(X, A)$ induced by corresponding maps $C_{p+1}(X) \to C_p(X)$. Then $\mathcal{C}^\bullet(X, A)$ is a complex and we have

$$H^p(X, A) = \mathcal{H}^p\big(\mathcal{C}^\bullet(X, A)\big). \tag{1}$$

Note that if $f\colon Y \to X$ is a continuous map, then we have a morphism of complexes $\mathcal{C}^\bullet(X, A) \to \mathcal{C}^\bullet(Y, A)$.

Since $A$ is fixed, we will denote by $\mathcal{C}_X^p$ the presheaf that associates to an open subset of $X$ the Abelian group $\mathcal{C}^p(U, A)$, with the restriction map corresponding to $U \subseteq V$ given by the map $\mathcal{C}_X^p(V, A) \to \mathcal{C}_X^p(U, A)$ induced by the inclusion. It is clear that we have a complex $\mathcal{C}_X^\bullet$ of presheaves on $X$. For every $p$, let $\mathcal{S}_X^p := (\mathcal{C}_X^p)^+$, so that we also have a complex $\mathcal{S}_X^\bullet$ of sheaves of $R$-modules on $X$. Note that we have a morphism of sheaves $A \to \mathcal{C}_X^0$ that associates to $s \in \Gamma(X, A)$, viewed as a locally constant function $X \to A$, the cocycle which associates to every 0-simplex in $A$, viewed as a point $x \in X$, the element $s(x) \in A$.

We claim that $A \to \mathcal{S}_X^\bullet$ is a resolution. Note first that if $U$ is a contractible open subset of $X$, then $H^p(U, A) = 0$ for all $p \geq 1$ and $H^0(U, A) = A$, hence $\Gamma(U, A) \to \Gamma(U, \mathcal{C}_X^\bullet)$ is a resolution. Since $X$ is locally contractible, we conclude that for every $x \in X$, at the level of stalks we have a resolution $A \to (\mathcal{C}_X^\bullet)_x = (\mathcal{S}_X^\bullet)_x$. This implies our claim.

If we are in a situation in which every open subset of $X$ is paracompact (for example, if $X$ is a topological manifold), then it is easy to deduce from Proposition 0.4 that each sheaf $\mathcal{S}_X^p$ is flasque. In general, we will show only that each sheaf $\mathcal{S}_X^p$ is soft, and the argument is a bit more involved. Note first that if $Y$ is any subspace of $X$, with $i\colon Y \hookrightarrow X$ being the inclusion map, then for every open subset $U$ of $X$, we have a canonical morphism of $R$-modules $\mathcal{C}^p(U,A) \to \mathcal{C}^p(U\cap Y, A)$. We thus obtain a morphism of presheaves $\mathcal{C}_X^p \to i_*\mathcal{C}_Y^p$ and thus a morphism of sheaves of $R$-modules $\mathcal{S}_X^p \to i_*\mathcal{S}_Y^p$. By the adjoint property of $(i^{-1}, i_*)$, this corresponds to a morphism of sheaves $\mathcal{S}_X^p|_Y \to \mathcal{S}_Y^p$. It is clear that if we restrict this to an open subset $V$ of $X$ that is contained in $Y$, then both sides are canonically isomorphic to $\mathcal{S}_V^p$ and the map is the identity.

We can now show that $\mathcal{S}_X^p$ is soft. Suppose that $Z$ is a closed subset of $X$ and $s \in \mathcal{S}_X^p(Z)$. By assertion i) in Lemma 2.3 in the write-up about soft sheaves, there is an open subset $U$ of $X$ containing $Z$, and $s_U \in \mathcal{S}_X^p(U)$ such that $s_U|_Z = s$. Let us choose an open subset $V$ of $X$, with $Z \subseteq V \subseteq \overline{V} \subseteq U$. Let $t \in \mathcal{S}_{\overline{V}}^p(\overline{V})$ be the image of $(s_U)|_{\overline{V}}$ via the morphism $\mathcal{S}_X^p|_{\overline{V}} \to \mathcal{S}_{\overline{V}}^p$. Since $X$ is paracompact, $\overline{V}$ is paracompact, too. It is straightforward to see that $\mathcal{C}_{\overline{V}}^p$ satisfies the hypothesis of Proposition 0.4: given an open cover $\overline{V} = \bigcup_{i\in I} U_i$ and cochains $\alpha_i \in \mathcal{S}_{\overline{V}}^p(U_i)$ such that $\alpha_i|_{U_i\cap U_j} = \alpha_j|_{U_i\cap U_j}$ for all $i$ and $j$, we define $\alpha \in \mathcal{S}_{\overline{V}}^p(\overline{V})$ such that for a $p$-simplex $\sigma$ in $\overline{V}$, we have $\alpha(\sigma) = \alpha_i(\sigma)$ if the image of $\sigma$ lies in some $U_i$, and 0 otherwise; it is clear that $\alpha$ is well-defined and $\alpha|_{U_i} = \alpha_i$ for all $i$. We conclude, using the proposition, that $t$ is the image of some $t' \in \mathcal{C}_{\overline{V}}^p(\overline{V})$. Since the map $\mathcal{C}_X^p(X) \to \mathcal{C}_{\overline{V}}^p(\overline{V})$ is clearly surjective, there is $s' \in \mathcal{C}_X^p(X)$ that maps to $t'$. Since $t|_V = s_U|_V$, it is straightforward to see that the image of $s'$ in $\mathcal{S}_X^p(X)$ restricts to $(s_U)|_V \in \mathcal{S}_X^p(V)$, and thus farther to $s \in \mathcal{S}_X^p(Z)$. This shows that $\mathcal{S}_X^p$ is soft.

We thus have a soft resolution $A \to \mathcal{S}_X^\bullet$ of sheaves of $R$-modules, hence Proposition 2.6 in the write-up about soft sheaves gives a canonical isomorphism

$$H^p(X,A) \simeq \mathcal{H}^p\big(\mathcal{S}_X^\bullet(X)\big). \tag{2}$$

Applying as above Proposition 0.4 for the sheaves $\mathcal{C}_X^p$, we see that for every $p$, we have a surjection

$$\mathcal{C}_X^p(X) \to \mathcal{S}_X^p(X).$$

Let $V^p$ be the kernel. This consists of the $p$-cochains $\beta$ with the property that there is some open cover $X = \bigcup_{i\in I} U_i$ such that $\beta$ vanishes on each $p$-simplex whose image is contained in some of the $U_i$. By considering the long exact sequence associated to the exact sequence of complexes

$$0 \to V^\bullet \to \mathcal{C}_X^\bullet(X) \to \mathcal{S}_X^\bullet(X) \to 0,$$

we see that if we show that $\mathcal{H}^p(V^\bullet) = 0$ for all $p$, then we are done by the isomorphisms (1) and (2),

By definition, $V^\bullet$ is the filtering direct limit of the complexes $V^\bullet(\mathcal{U})$, where $\mathcal{U} = (U_i)_{i\in I}$ is an open cover of $X$ and where $V^p(\mathcal{U})$ consists of the $p$-cochains that vanish on $\mathcal{U}$-*small* simplices, that is, $p$-simplexes in $X$ whose image is contained in some of the $U_i$. Since filtering direct limits form an exact functor, it is enough to show that $\mathcal{H}^p\big(V^\bullet(\mathcal{U})\big) = 0$ for all $\mathcal{U}$ and all $p$.

If $\mathcal{C}_p^{\mathcal{U}}(X)$ is the subgroup of $\mathcal{C}_p(X)$ generated by simplices whose image is contained in some open subset in $\mathcal{U}$, then $\mathcal{C}_\bullet^{\mathcal{U}}(X)$ is a subcomplex of $\mathcal{C}_\bullet(X)$. A basic result, proved using barycentric subdivisions, says that the inclusion

$$\mathcal{C}^{\mathcal{U}}_{\bullet}(X) \hookrightarrow \mathcal{C}_{\bullet}(X)$$

is a homotopy equivalence[2] (see [Hat02, Proposition 2.21]). In this case, applying $\mathrm{Hom}_{\mathbf{Z}}(-, A)$ gives a homotopy equivalence

$$u\colon \mathcal{C}^{\bullet}(X, A) \to \mathrm{Hom}_{\mathbf{Z}}(\mathcal{C}^{\mathcal{U}}_{\bullet}, A),$$

which thus induces isomorphisms in cohomology. On the other hand, $u$ is a surjective morphism of complexes, whose kernel is equal to $V^{\bullet}(\mathcal{U})$. By considering the corresponding long exact sequence in cohomology, we conclude that $\mathcal{H}^p\big(V^{\bullet}(\mathcal{U})\big) = 0$ for all $p$. This completes the proof of the theorem. □

## References

[Hat02] A. Hatcher, Algebraic topology, Cambridge University Press, Cambridge, 2002. 1, 4

[2] This means that there is a morphism of complexes in the opposite direction such that both compositions are homotopic to the respective identity maps.