Problem session 9

While we will not mention this explicitly in what follows, all schemes are assumed to be of finite type over an algebraically closed field k.

Problem 1. Show that if $(X, \mathcal{O})$ is a reduced scheme and if $U \subseteq X$ is a dense open subset, then the restriction map $\mathcal{O}(X) \to \mathcal{O}(U)$ is injective.

Problem 2. Let $(X, \mathcal{O})$ be a scheme and $f \in \mathcal{O}(X)$. Recall that we have a corresponding continuous function $\widetilde{f}\colon X \to k$, such that $\widetilde{f}(x)$ is the class of $f_x \in \mathcal{O}_x$ in the quotient modulo the maximal ideal, which is canonically isomorphic to k. Let

$$X_f := \{x \in X \mid \widetilde{f}(x) \neq 0\}.$$

i) Describe this set when $X = \mathrm{Specm}(R)$.
ii) Show that the restriction map induces a morphism of k-algebras $\mathcal{O}(X)_f \to \mathcal{O}(X_f)$. Prove that for every X, this is an isomorphism.

Problem 3. For an arbitrary scheme X, use the previous problem and the canonical morphism

$$X \to \operatorname{Spec} \mathcal{O}(X)$$

to prove the following criterion for X to be affine: if $f_1, \ldots, f_r \in \mathcal{O}(X)$ are such that they generate the unit ideal in $\mathcal{O}(X)$ and all X_{f_i} are affine schemes, then X is affine.

Problem 4.

i) Show that if $X_1, \ldots, X_n$ are schemes, then on the disjoint union $\bigsqcup_{i=1}^n X_i$ there is a unique scheme structure (up to a canonical isomorphism) such that each inclusion $X_i \subset X$ gives an open immersion .
ii) Show that for every scheme X, its connected components are open in X.
iii) Show that $\mathrm{Specm}(R)$ is disconnected if and only if there is an isomorphism $R \simeq R_1 \times R_2$ for suitable k-algebras R_1 and R_2.