Problem session 8

As usual, all schemes are assumed to be of finite type over an algebraically closed field k.

Problem 1. Blow-ups appear naturally when resolving indeterminacies of rational maps, as follows. Suppose that $\mathcal{L}$ is a line bundle on an integral scheme X, and $V \subseteq \Gamma(X, \mathcal{L})$ is a finite dimensional linear subspace, defining the rational map $\varphi = \varphi_V \colon X \dashrightarrow \mathbf{P}(V)$.

i) Show that if Z is the base locus of V (with the corresponding scheme structure), and if $\pi \colon \mathrm{Bl}_Z X \to X$ is the blow-up of X along Z, then the rational map $\varphi \circ \pi^{-1}$ is in fact a morphism.

ii) In general, if $h \colon X \dashrightarrow Y$ is a rational map between the integral schemes X and Y, with Y separated, then the *graph* Γ_h of h is defined as follows: if $U \subseteq X$ is an open subset of X on which h is defined, then Γ_h is the closure in $X \times Y$ of the graph of $h \colon U \to Y$ (check that this definition is independent of U). Show that if φ is as above, then $\mathrm{Bl}_Z(X)$ is isomorphic to the graph of φ.

Problem 2. If Y is a scheme, and $y \in Y$ is a point defined by $\mathfrak{m}_y$, then the (abstract) *tangent cone* of Y at y is $C_yY := \mathrm{Spec}(\oplus_{i\geq 0}\mathfrak{m}^i/\mathfrak{m}^{i+1})$, and the projectivized tangent cone of Y at y is $\mathbf{P}(C_yY) := \mathrm{Proj}(\oplus_{i\geq 0}\mathfrak{m}^i/\mathfrak{m}^{i+1})$ (hence $\mathbf{P}(C_yY)$ is isomorphic to the inverse image of y in $\mathrm{Bl}_y(Y)$). Suppose now that Y is a closed subscheme of $X = \mathbf{A}^n$ (more generally, a similar discussion holds if we only assume X nonsingular).

i) Show that we have a closed immersion $\mathbf{P}(C_yY) \hookrightarrow \mathbf{P}(C_y\mathbf{A}^n) \simeq \mathbf{P}^{n-1}$. The affine cone over the image is the *embedded tangent cone* to Y at y.

ii) Show that the tangent cone of Y at y has dimension equal to $\dim(\mathcal{O}_{Y,y})$.

iii) Show that if Y is a hypersurface in $\mathbf{A}^n$ defined by $(f = 0)$, and if $f = f_m + f_{m+1} + \ldots + f_d$, with $\deg(f_i) = i$ and $f_m \neq 0$, then C_0Y is defined in $\mathbf{A}^n$ by the ideal (f_m).

iv) Show that more generally, if for f as above we put $\mathrm{in}(f) = f_m$, then for every closed subscheme Y of $\mathbf{A}^n$ the ideal defining C_0Y is generated by those $\mathrm{in}(f)$ for all f in the ideal defining Y (note: it is not enough to only consider a system of generators of the ideal defining Y).

v) Show that the embedded tangent cone of Y at y is contained in the tangent space of Y at y.