Problem session 8

While we will not mention this explicitly in what follows, all schemes are assumed to be of finite type over an algebraically closed field k.

Problem 1. Let X and Y be two locally ringed spaces over k, and let $X = U_1 \cup \ldots \cup U_r$ be an open cover of X.

i) Show that if $f, g\colon X \to Y$ are two morphisms such that $f|_{U_i} = g|_{U_i}$ for all i, then $f = g$ (for any open subset U of X, and any morphism $h\colon X \to Y$, we denote by $h|_U$ the composition of h with the morphism $i\colon U \to X$ induced by inclusion).
ii) Show that if we have morphisms $h_i\colon U_i \to Y$ such that $h_i|_{U_i \cap U_j} = h_j|_{U_i \cap U_j}$ for all i and j, then there is a unique morphism $h\colon X \to Y$ such that $h|_{U_i} = h_i$ for every i.

Problem 2.

i) Let X be a scheme, and $i\colon W \to X$ an open immersion. Show that if $f\colon Y \to X$ is a morphism of schemes such that $f(Y) \subseteq i(W)$, then there is a unique morphism of schemes $g\colon Y \to W$ such that $i \circ g = f$.
ii) Deduce that if $f\colon Y \to X$ is a morphism of schemes, and if U is an open subscheme of X, then we have an induced morphism of schemes $f^{-1}(U) \to U$.

Problem 3. Let $f\colon X \to Y$ be a morphism of schemes. If there is an open cover $Y = V_1 \cup \ldots \cup V_r$ such that the induced morphism $f^{-1}(V_i) \to V_i$ is an isomorphism for every i, then f is an isomorphism.

Problem 4. Show that if X is an affine scheme, then for every scheme Y the canonical map

$$\mathrm{Hom}(Y, X) \to \mathrm{Hom}_{k-\mathrm{alg}}(\mathcal{O}(X), \mathcal{O}(Y))$$

is a bijection. (You may assume you know this when also Y is affine, as we will prove this in class).