Problem session 7

As usual, all schemes are assumed to be of finite type over an algebraically closed field k.

Problem 1. Show that if X is a reduced scheme, then X is affine if and only if each irreducible component of X is affine.

Problem 2. Prove the following theorem of Chevalley: if $f\colon X \to Y$ is a finite surjective morphism of schemes, and X is affine, then Y is affine.

Hint: use the following steps:

i) Reduce to the case when both X and Y are integral schemes.
ii) Show that if X and Y are integral, then there is a coherent sheaf $\mathcal{F}$ on X, and a morphism of sheaves $f\colon \mathcal{O}_Y^{\oplus r} \to f_*(\mathcal{F})$ for some $r \geq 1$, such that f is an isomorphism over an open subset of Y.
iii) Deduce that under the assumptions in ii), given a coherent sheaf $\mathcal{N}$ on Y, there is a coherent sheaf $\mathcal{M}$ on X and a morphism $f_*(\mathcal{M}) \to \mathcal{N}^{\oplus r}$ that is an isomorphism over an open subset of Y.
iv) Prove Chevalley's theorem by Noetherian induction.

Problem 3. Let X be a scheme, and U an open subscheme. Prove the following assertions:

i) If $\mathcal{F}$ is a coherent sheaf on U, then there is a coherent sheaf $\mathcal{G}$ on X such that $\mathcal{G}|_U \simeq \mathcal{F}$.
ii) Furthermore, if $\mathcal{M}$ is a coherent sheaf on X such that $\mathcal{F}$ is a subsheaf of $\mathcal{M}|_U$, then we may take $\mathcal{G}$ to be a subsheaf of $\mathcal{M}$.

Hint: consider the following intermediate steps:

a) Show that if X is an *affine* scheme and $\mathcal{P}$ is a quasicoherent sheaf on X, and if $(\mathcal{P}_i)_i$ is the family of coherent subsheaves of $\mathcal{P}$, then for every open subset U of X we have $\mathcal{P}(U) = \bigcup_i \mathcal{P}_i(U)$.
b) Prove i) and ii) above when X is affine.
c) Prove ii), and then i) above in the general case.