Problem session 7

Problem 1. Let $\mathcal{F}$ and $\mathcal{F}'$ be two subsheaves of the sheaf $\mathcal{G}$. Show that if $\mathcal{F}_x = \mathcal{F}'_x$ for every $x \in X$, then $\mathcal{F} = \mathcal{F}'$.

Problem 2. Give an example of a surjective morphism $\mathcal{F} \to \mathcal{G}$ of sheaves of abelian groups on X, such that the induced morphism $\mathcal{F}(X) \to \mathcal{G}(X)$ is not surjective.

Problem 3. Let $f\colon X \to Y$ be a continuous map. Show that the pair of functors (f^{-1}, f_*) is an adjoint pair, that is, for every sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y, there is a functorial isomorphism

$$\mathrm{Hom}(f^{-1}(\mathcal{G}), \mathcal{F}) \simeq \mathrm{Hom}(\mathcal{G}, f_*(\mathcal{F})).$$

Note that as a consequence of this isomorphism we have canonical morphisms $\mathcal{G} \to f_*(f^{-1}(\mathcal{G}))$ and $f^{-1}(f_*(\mathcal{F})) \to \mathcal{F}$.

Problem 4. Show that if

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is a short exact sequence of sheaves of abelian groups on X, then for every open subset $U \subseteq X$, the induced sequence

$$0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U)$$

is exact.

Problem 5. Let $f\colon X \to Y$ be a continuous map. We denote by Sh_X and Sh_Y the categories of sheaves of abelian groups on X and Y, respectively. Show that the functor $f_*\colon \mathrm{Sh}_X \to \mathrm{Sh}_Y$ is left exact, while the functor $f^{-1}\colon \mathrm{Sh}_Y \to \mathrm{Sh}_X$ is exact.