Problem session 6

As usual, all schemes are assumed to be of finite type over an algebraically closed field k.

Problem 1. Show that if X is an integral projective scheme, then $\Gamma(X, \mathcal{O}_X) = k$.

Problem 2. Show that if L is an invertible sheaf on a projective integral scheme X over the field k, such that $H^0(X, L) \neq 0$ and $H^0(X, L^{-1}) \neq 0$, then $L \simeq \mathcal{O}_X$. Deduce that if $X \subseteq \mathbf{P}^n$ is a closed integral subscheme of positive dimension, then $H^0(X, \mathcal{O}_X(-m)) = 0$ for every $m > 0$.

Problem 3. Let X be a normal integral scheme and U an open subset of X.

i) Show that if $Y_1, \ldots, Y_r$ are the irreducible components of $X \smallsetminus U$ that have codimension one in X, then there is an exact sequence

$$\mathbb{Z}^r \xrightarrow{f} \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 0,$$

where $f(a_1, \ldots, a_r)$ is the class of $\sum_i a_i Y_i$.

ii) Deduce that if H is a prime divisor in $\mathbf{P}^n$ of degree d, then $\mathrm{Cl}(\mathbf{P}^n \smallsetminus H) \simeq \mathbb{Z}/d\mathbb{Z}$.

Problem 4. Show that if X is a reduced scheme, then X is affine if and only if each irreducible component of X is affine.

Problem 5. Prove the following theorem of Chevalley: if $f\colon X \to Y$ is a finite surjective morphism of separated schemes, and X is affine, then Y is affine.

Hint: use the following steps:

i) Reduce to the case when both X and Y are integral schemes.
ii) Show that if X and Y are integral, then there is a coherent sheaf $\mathcal{F}$ on X, and a morphism of sheaves $f\colon \mathcal{O}_Y^{\oplus r} \to f_*(\mathcal{F})$ for some $r \geq 1$, such that f is an isomorphism over an open subset of Y.
iii) Deduce that under the assumptions in ii), given a coherent sheaf $\mathcal{N}$ on Y, there is a coherent sheaf $\mathcal{M}$ on X and a morphism $f_*(\mathcal{M}) \to \mathcal{N}^{\oplus r}$ that is an isomorphism over an open subset of Y.
iv) Prove Chevalley's theorem by Noetherian induction.