Problem session 6

Problem 1. Let $f\colon X \to Y$ be an arbitrary morphism of (quasi-affine) varieties. For every $x \in X$, we put

$$e(x) := \max\{\dim(Z) \mid Z = \text{irreducible component of } f^{-1}(f(x)),\, x \in Z\}.$$

Show that the function $x \to e(x)$ is upper semicontinuous, that is, for every $m \in \mathbb{Z}$, the set $\{x \in X \mid e(x) \geq m\}$ is closed in X.

Problem 2. Let $f\colon X \to Y$ be any morphism of (quasi-affine) varieties. One can ask whether the function $Y \to \mathbb{Z}$, that takes y to $\dim(f^{-1}(y))$ is upper semi-continuous (recall our convention that $\dim(\emptyset) = -1$). We will see later that this is the case for the so-called *proper morphisms*. However, show that this is not true in general: given any nonnegative integers $r < s$, give an example of a morphism $f\colon X \to Y$ such that for some $y_0 \in Y$ we have $\dim(f^{-1}(y_0)) = r$, and $\dim(f^{-1}(y)) = s$ for every $y \neq y_0$.

Problem 3. (Automorphisms of $\mathbf{A}^n$).

i) Give examples of automorphisms of $\mathbf{A}^n$.
ii) Let $f\colon \mathbf{A}^n \to \mathbf{A}^n$ be a morphism defined by $f_1, \ldots, f_n \in k[x_1, \ldots, x_n]$. Denote by $J(f) := \det(\partial f_i/\partial x_j)$ the determinant of the Jacobian matrix of f. Show that if f is an automorphism, then $J(f)$ is a nonzero element of k.

Remark. The converse of the assertion in ii) is a famous open problem, the *Jacobian Conjecture*. It is open even in the case $n = 2$.

Problem 4. Suppose that $\mathrm{char}(k) = p > 0$, and let $X \subseteq \mathbf{A}^n$ be a closed subset. We say that X *is defined over* the finite field $\mathbf{F}_q$ (where $q = p^e$) if the ideal $I(X)$ of X can be generated by polynomials in $\mathbf{F}_q[x_1, \ldots, x_n]$. Recall that $F\colon \mathbf{A}^n \to \mathbf{A}^n$ is the Frobenius morphism given by $F(u_1, \ldots, u_n) = (u_1^p, \ldots, u_n^p)$.

i) Show that if X is defined over $\mathbf{F}_q$, with $q = p^e$, then F^e induces a morphism $\mathrm{Frob}_{X,e}\colon X \to X$.
ii) Show that $\mathrm{Frob}_{X,e}$ is a finite surjective morphim.
iii) Show that the fixed points of $\mathrm{Frob}_{X,e}$ are the $\mathbf{F}_q$*-points of* X, that is, the points of X that lie in $\mathbf{F}_q^n \subseteq k^n$.

Problem 5. Show that every positive-dimensional variety over k has the same cardinality as k. Deduce that any two irreducible curves over k are homeomorphic (a *curve* is a variety of pure dimension one).