Problem session 5

Problem 1. Let $X = \mathrm{Projm}(R)$, and $M = \oplus_{i\in\mathbb{Z}} M_i$ a finitely generated graded R-module. Show that $\widetilde{M} = 0$ if and only if there is d_0 such that $M_d = 0$ for all $d \geq d_0$.

Problem 2. Let I be a homogeneous ideal in $R = A[x_0, \ldots, x_n]$, defining the closed subscheme $X \hookrightarrow \mathbf{P}^n_A$. Show that the induced morphism

$$(R/I)_d \to \Gamma(X, \mathcal{O}_X(d))$$

is an isomorphism for $d \gg 0$.

Problem 3. Let L be a line bundle on a scheme X. Show that if m is a positive integer, then L is ample if and only if L^m is ample.

Problem 4. Let $X = \mathrm{Projm}(R)$, and let

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \ldots \to \mathcal{F}_r \to 0$$

be an exact complex of coherent sheaves on X. Show that there is m_0 such that the induced complex

$$0 \to \Gamma(X, \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{O}(m)) \to \Gamma(X, \mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{O}(m)) \to \ldots \to \Gamma(X, \mathcal{F}_m \otimes_{\mathcal{O}_X} \mathcal{O}(m)) \to 0$$

is exact for every $m \geq m_0$.

Problem 5. Let X be a scheme, and Y a closed subscheme of X defined by the ideal sheaf $\mathcal{J}$. Show that if $\mathcal{J}^2 = 0$, then there is an exact sequence

$$H^1(X, \mathcal{J}) \to \mathrm{Pic}(X) \to \mathrm{Pic}(Y) \to H^2(X, \mathcal{J}).$$

Deduce in particular that if X is affine, then we have an isomorphism $\mathrm{Pic}(X) \simeq \mathrm{Pic}(Y)$.