# Problem session 5

**Problem 1**. Let $f\colon X \to Y$ be a morphism of algebraic varieties. Suppose that $Y$ is irreducible, and that all fibers of $f$ are irreducible, of the same dimension $d$ (in particular, $f$ is surjective).

i) There is a unique irreducible component of $X$ that dominates $Y$.
ii) Every irreducible component $X_i$ of $X$ is a union of fibers of $f$. Its dimension is equal to $\dim(\overline{f(X_i)}) + d$.

In particular, we can conclude that $X$ is irreducible if either of the following holds:

a) $X$ is pure-dimensional.
b) $f$ is closed, that is, for every $W$ closed in $X$, its image $f(W)$ is closed in $Y$.

Let $X$ be a topological space. A subset of $X$ is *constructible* if it can be written as a finite union of locally closed subsets of $X$.

**Problem 2**. Let $X$ be a Noetherian topological space.

i) Show that the set of constructible subsets of $X$ is closed under finite unions, finite intersections, and taking complements (it is the smallest such set containing all open subsets of $X$).
ii) Show that if $W$ is constructible in $X$, then there is $U \subseteq W$, such that $U$ is open in $\overline{W}$ (in particular, $U$ is a locally closed subset of $X$).

The importance of the above notion comes from the following theorem of Chevalley, that we will prove in class.

**Theorem**. If $f\colon X \to Y$ is any morphism of algebraic varieties, then for every constructible subset $W \subseteq X$, its image $f(W)$ is constructible. In particular, $f(X)$ is constructible.

**Problem 3**. Let $G$ be a linear algebraic group, acting algebraically on a variety $X$. Show that for every point $p \in X$, its orbit $G \cdot p$ is a locally closed subset of $X$.

**Problem 4**. Show that if $f\colon X \to Y$ is a birational morphism of (quasi-affine) algebraic varieties, then there is an open subset $W \subseteq Y$ such that the induced map $f^{-1}(W) \to W$ is an isomorphism.