Problem session 4

Problem 1. A *linear algebraic group* is an affine variety G that is also a group, and such that the multiplication $\mu\colon G \times G \to G$, $\mu(g,h) = gh$, and the inverse map $\iota\colon G \to G$, $\iota(g) = g^{-1}$ are morphisms of algebraic varieties. A morphism of linear algebraic groups is a morphism of algebraic varieties, that is also a group homomorphism.

i) Show that $(k, +)$ and $(k^*, \cdot)$ are linear algebraic groups.
ii) Show that the set $\mathrm{GL}_n(k)$ of $n \times n$ invertible matrices with coefficients in k is a linear algebraic group.
iii) Show that if G and H are linear algebraic groups, then the product $G \times H$ has an induced structure of linear algebraic group. In particular, the (algebraic) *torus* $(k^*)^n$ is a linear algebraic group with respect to component-wise multiplication.

Problem 2. Let G be a linear algebraic group acting algebraically on an affine variety X (that is, the map $G \times X \to X$ given by the action is a morphism of algebraic varieties). Show that in this case G has an induced linear action on $\mathcal{O}(X)$. While $\mathcal{O}(X)$ has infinite dimension over k, show that the action of G on $\mathcal{O}(X)$ has the following finiteness property: every element $f \in \mathcal{O}(X)$ lies in some finite-dimensional vector subspace V of $\mathcal{O}(X)$ that is preserved by the G-action (Hint: consider the corresponding k-algebra homomorphism $\mathcal{O}(X) \to \mathcal{O}(G) \otimes_k \mathcal{O}(X)$).

Problem 3. Let G and X be as in the previous problem. Consider a system of k-algebra generators $f_1, \ldots, f_m$ of $\mathcal{O}(X)$, and apply the previous problem to each of these elements to show that there is a morphism of algebraic groups $G \to \mathrm{GL}_N(k)$, and an isomorphism of X with a closed subset of $\mathbb{A}^N$, such that the action of G on X is induced by the standard action of $\mathrm{GL}_N(k)$ on $\mathbb{A}^N$. Use the same argument to show that every linear algebraic group is isomorphic to a closed subgroup of some $\mathrm{GL}_N(k)$.

Problem 4. Show that the linear algebraic group $\mathrm{GL}_m(k) \times \mathrm{GL}_n(k)$ has an algebraic action on the space $M_{m,n}(k)$ (identified to $\mathbb{A}^{mn}$), induced by left and right matrix multiplication. What are the orbits of this action ? Show that the orbits are locally closed subsets of $M_{m,n}(k)$. N.B. This is a general fact about algebraic group actions.