Problem session 3

Problem 1. Let X be a scheme. A *family of transition functions* $(U_i, \phi_{i,j})$ on X is given by a (finite) open cover $X = \bigcup_{i\in I} U_i$, and by a family $(\phi_{i,j})_{i,j\in I}$, where $\phi_{i,j} \in \mathcal{O}(U_i \cap U_j)$ is invertible, satisfying the following "cocycle condition":

$$\phi_{i,j} \cdot \phi_{j,k} = \phi_{i,k} \text{ on } U_i \cap U_j \cap U_k$$

for every i, j, and k. We have seen in class that given a family of transition functions as above, we get an associated line bundle on X (unique up to isomorphism), and every line bundle on X arises this way.

i) We say that an open cover $X = \bigcup_{\alpha\in J} W_\alpha$ is a refinement of the cover given by $(U_i)_{i\in I}$ if there is a map $\rho\colon J \to I$ such that $W_\alpha \subseteq U_{\rho(\alpha)}$ for every $\alpha \in J$. In this case, if $(\phi_{i,j})_{i,j\in I}$ is as above, then we get an induced family of transition functions by taking $(\psi_{\alpha\beta})_{\alpha,\beta\in J}$, with $\psi_{\alpha,\beta} = \phi_{\rho(\alpha),\rho(\beta)}|_{W_\alpha\cap W_\beta}$. Show that the two families $(U_i, \phi_{i,j})$ and $(W_\alpha, \psi_{\alpha,\beta})$ define isomorphic line bundles. In particular, whenever having two families of transition functions, we may assume that the corresponding open covers are the same.

ii) Let $(U_i, \phi_{i,j})$ and $(U_i, \psi_{i,j})$ be two families of transition functions on X, defining the line bundles $\mathcal{L}$ and $\mathcal{L}'$. Show that the family of transition functions $(U_i, \phi_{i,j}\psi_{i,j})$ defines $\mathcal{L} \otimes \mathcal{L}'$, and the family of transition functions $(U_i, \phi_{i,j}^{-1})$ defines $\mathcal{L}^{-1}$.

iii) Show that the family of transition functions $(U_i, \phi_{i,j})$ defines the trivial line bundle $\mathcal{O}_X$ if and only if it is a "coboundary", that is, there are invertible functions $f_i \in \mathcal{O}(U_i)$ for every i such that $\phi_{i,j} = f_i|_{U_i\cap U_j} \cdot f_j^{-1}|_{U_i\cap U_j}$.

Problem 2. Let X be a closed subset of $\mathbb{P}^n$. The *Fano variety of lines* on X consists of the lines $\ell \in G(2, n+1)$ such that $\ell \subseteq X$. Show that this is a closed subset of $G(2, n+1)$. Can you describe the Fano variety of lines for the quadric $xy - zw = 0$ in $\mathbb{P}^3$?

Problem 3. Let V be a vector space over k of dimension n, and $1 \le \ell_1 < \ldots < \ell_r \le n$. A *flag of type* $(\ell_1, \ldots, \ell_r)$ in V is a sequence of linear subspaces $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_r \subseteq V$, with $\dim_k(V_i) = \ell_i$.

i) Show that the set

$$\mathrm{Fl}_{\ell_1,\ldots,\ell_r}(V) := \{(V_1, \ldots, V_r) \in G(\ell_1, V) \times \cdots \times G(\ell_r, V) \mid V_1 \subseteq \cdots \subseteq V_r\}$$

is a closed subset in the product of Grassmanians. In particular, this is a projective variety that parametrizes flags in V of type $(\ell_1, \ldots, \ell_r)$.

ii) Show that the projection on the last component gives a surjective morphism $\mathrm{Fl}_{\ell_1,\ldots,\ell_r}(V) \to G(\ell_r, V)$, such that each fiber is isomorphic to $\mathrm{Fl}_{\ell_1,\ldots,\ell_{r-1}}(k^{\ell_r})$.

iii) Use induction on r to prove that each flag variety $\mathrm{Fl}_{\ell_1,\ldots,\ell_r}(V)$ is irreducible, of dimension

$$\sum_{i=1}^{r} \ell_i(\ell_{i+1} - \ell_i)$$

(where we put $\ell_{r+1} = n$). In particular, the dimension of the complete flag variety $\mathrm{Fl}(V)$ on V (this is the case $\ell_i = i$ for $1 \leq i \leq r = n-1$) is $\frac{n(n-1)}{2}$.