Problem session 2

Problem 1. Show that if X and Y are topological spaces, with X irreducible, and $f\colon X \to Y$ is a continuous map, then $\overline{f(X)}$ is irreducible. Use this to show that the closed subset

$$M^r_{m,n}(k) = \{A \in M_{m,n}(k) \mid \operatorname{rank}(A) \leq r\}$$

of $\mathbf{A}^{mn}$ is irreducible.

Problem 2. Let X be a topological space.

i) Show that if Y is a subset of X (with the induced topology), then Y is irreducible if and only if its closure $\overline{Y}$ is irreducible.
ii) Suppose that X is Noetherian, and that Y is a subset X. Show that if $Y = Y_1 \cup \ldots \cup Y_r$ is the irreducible decomposition of Y, then $\overline{Y} = \overline{Y_1} \cup \ldots \cup \overline{Y_r}$ is the irreducible decomposition of $\overline{Y}$.

Problem 3.

i) Show that $\mathbf{A}^1 \smallsetminus \{0\}$ is an affine variety (recall: this means that it is isomorphic to a closed subset of an affine space).
ii) Let $U = \mathbf{A}^2 \smallsetminus \{(0,0)\}$. What is $\mathcal{O}(U)$?
iii) Deduce that U is not an affine variety.

Problem 4. Show that the set

$$B = \left\{(a,b,c,d) \in \mathbf{A}^4 \mid \operatorname{rank}\begin{pmatrix} a & b & c \\ b & c & d \end{pmatrix} \leq 1\right\}$$

is an irreducible closed subset of $\mathbf{A}^4$. Determine its ideal.