Problem session 11

While we will not mention this explicitly in what follows, all schemes are assumed to be of finite type over an algebraically closed field k.

Problem 1. Show that if $f\colon X \to Y$ is a proper morphism of schemes, then we have the following semicontinuity statement for fiber dimensions: for every d, the set

$$\{y \in Y \mid \dim(f^{-1}(y)) \geq d\}$$

is closed in Y. Compare to the statement we had for arbitrary morphisms.

Problem 2. Show that both notions of separated morphisms and proper morphisms are *local on the base*, in the following sense: given a morphism $f\colon X \to Y$ and an open cover $Y = \bigcup_i W_i$ such that each induced morphism $f^{-1}(W_i) \to W_i$ has the given property, then so does f.

Problem 3. Show that if X is an affine scheme, then X is complete if and only if it is zero-dimensional.

Problem 4. Let $f\colon X \to Y$ be a morphism of algebraic varieties, with X complete, and Y affine. Show that f is constant on each connected component of X.

Problem 5. A *linear subspace* of $\mathbf{P}^n$ is a closed subscheme of $\mathbf{P}^n$ defined by an ideal generated by homogeneous polynomials of degree one. A *hyperplane* is a linear subspace of codimension one.

i) Show that if L is a linear subspace in $\mathbf{P}^n$ of dimension r, then there is an isomorphism $L \simeq \mathbf{P}^r$. Note, in particular, that every linear subspace in $\mathbf{P}^n$ is reduced and irreducible (hence we identify X with its support).
ii) Show that a closed subset Y of $\mathbf{P}^n$ is a linear subspace if and only if the affine cone $C(Y) \subseteq \mathbf{A}^{n+1}$ is a linear subspace.
iii) Show that the hyperplanes in $\mathbf{P}^n$ are in bijection with the points of "another" projective space $\mathbf{P}^n$, usually denoted by $(\mathbf{P}^n)^*$.