Problem session 10

In the first problem we consider the *Segre embedding* to show that the product of two projective schemes is again projective.

Problem 1. Consider two projective spaces $\mathbf{P}^m$ and $\mathbf{P}^n$. Let $N = (m+1)(n+1) - 1$, and we denote the coordinates on $\mathbf{A}^{N+1}$ by $z_{i,j}$, with $0 \leq i \leq m$ and $0 \leq j \leq n$.

1) Show that the map $\mathbf{A}^{m+1} \times \mathbf{A}^{n+1} \to \mathbf{A}^{N+1}$ given by
$$((x_i)_i, (y_j)_j) \to (x_i y_j)_{i,j}$$
induces a morphism
$$\phi_{m,n} \colon \mathbf{P}^m \times \mathbf{P}^n \to \mathbf{P}^N.$$
2) Consider the ring homomorphism $f_{m,n} \colon k[z_{i,j} \mid 0 \leq i \leq m, 0 \leq j \leq n] \to k[x_1, \ldots, x_m, y_1, \ldots, y_n]$, given by $f_{m,n}(z_{i,j}) = x_i y_j$. Show that $\ker(f_{m,n})$ is a homogeneous prime ideal that defines in $\mathbf{P}^N$ the image of $\phi_{m,n}$ (in particular, this image is closed).
3) Show that $\phi_{m,n}$ is a closed immersion.
4) Deduce that if X and Y are (quasi)projective varieties, then $X \times Y$ is a (quasi)projective variety.

Problem 2. Let $R = \bigoplus_{m \geq 0} R_m$ be a graded k-algebra that is generated as a k-algebra by finitely many elements in R_1. Consider $X = \mathrm{Projm}(R)$, and for every homogeneous $f \in R$ of positive degree, let $D_+(f) = \{\mathfrak{p} \in X \mid f \notin \mathfrak{p}\}$. Show that we have an isomorphism of schemes $D_+(f) \simeq \mathrm{Specm}(R_{(f)})$.

Problem 3. Recall that $GL_{n+1}(k)$ denotes the set of invertible $(n+1) \times (n+1)$ matrices with entries in k. Let $PGL_n(k)$ denote the quotient $GL_{n+1}(k)/k^*$, where k^* acts by
$$\lambda \cdot (a_{i,j}) = (\lambda a_{i,j}).$$

i) Show that $PGL_n(k)$ has a natural structure of linear algebraic group, and that it is irreducible.
ii) Prove that $PGL_n(k)$ acts algebraically on $\mathbf{P}^n$.
iii) Show that given two sets of points in $\mathbf{P}^n$
$$\Gamma = \{P_0, \ldots, P_n\} \text{ and } \Gamma' = \{Q_0, \ldots, Q_n\},$$
neither of them being contained in a hyperplane, there is a unique $A \in PGL_n(k)$ such that $A \cdot P_i = Q_i$ for every i (a *hyperplane* in $\mathbf{P}^n$ is a closed subset defined by a nonzero linear polynomial).

Two subsets of $\mathbf{P}^n$ are *projectively equivalent* if they differ by an automorphism as above (we will see later that these are, indeed, all automorphisms of $\mathbf{P}^n$).