Problem session 2

Problem 1. Let $\mathcal{F}$ be a coherent sheaf on a scheme X. Show that $\mathcal{F}$ is locally free of rank r if and only if $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module of rank r for every $x \in X$.

Probem 2. Let $f\colon X \to Y$ be a morphism of schemes. Show that if $\mathcal{F}_1$ and $\mathcal{F}_2$ are $\mathcal{O}_Y$-modules, then there is a canonical isomorphism

$$f^*(\mathcal{F}_1 \otimes_{\mathcal{O}_Y} \mathcal{F}_2) \simeq f^*(\mathcal{F}_1) \otimes_{\mathcal{O}_X} f^*(\mathcal{F}_2).$$

Prove a similar statement for symmetric and exterior products.

Problem 3. Let $f\colon X \to Y$ be a morphism of schemes, and let $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module of rank r. Let $V(\mathcal{E}) = \mathcal{S}pec(\mathrm{Sym}(\mathcal{E}))$ be the geometric vector bundle associated to $\mathcal{E}$.

i) Show that $f^*(\mathcal{E})$ is locally free of rank r.
ii) Show that there is a Cartesian diagram

$$\begin{array}{ccc} V(f^*(\mathcal{E})) & \longrightarrow & V(\mathcal{E}) \\ \Big\downarrow \psi & & \Big\downarrow \phi \\ X & \xrightarrow{\ f\ } & Y \end{array}$$

iii) Deduce that for every $y \in Y$, there is an isomorphism $\phi^{-1}(y) \simeq (\mathcal{E}_y/\mathfrak{m}_y\mathcal{E}_y)^\vee$, where $\mathfrak{m}_y$ denotes the maximal ideal in $\mathcal{O}_{Y,y}$, and $W^\vee$ denotes the dual of a vector space W.

Problem 4. Let X be an integral scheme, and $\mathcal{F}$ a coherent sheaf on X. Show that $\mathcal{F}$ is locally free of rank r if and only if for every $y \in Y$, we have $\dim_{k(y)}(\mathcal{F}_y/\mathfrak{m}_y\mathcal{F}_y) = r$, where $\mathfrak{m}_y$ is the maximal ideal in $\mathcal{O}_{Y,y}$, and $k(y) \simeq k$ is the residue field.

Problem 5. Let X be a scheme, and $\mathcal{E}$, $\mathcal{F}$ locally free $\mathcal{O}_X$-modules (of finite rank). A morphism of sheaves $\phi\colon \mathcal{E} \to \mathcal{F}$ is a *morphism of vector bundles* if $\mathrm{Coker}(\phi)$ is locally free.

i) Show that if ϕ is a morphism of vector bundles, then $\ker(\phi)$ and $\mathrm{Im}(\phi)$ are also locally free.
ii) Suppose that X is an integral scheme. Show that f is a morphism of vector bundles if and only if there is s such that for every $y \in Y$, the induced morphism of $k(y)$-vector spaces

$$\mathcal{E}_y/\mathfrak{m}_y\mathcal{E}_y \to \mathcal{F}_y/\mathfrak{m}_y\mathcal{F}_y$$

has rank s.