An irreducibility criterion

This is a solution of the first problem on Pb. set 5.

Proposition. Let $f\colon X \to Y$ be a morphism of algebraic varieties. Suppose that Y is irreducible, and that all fibers of f are irreducible, of the same dimension d (in particular, f is surjective).

i) There is a unique irreducible component of X that dominates Y.
ii) Every irreducible component X_i of X is a union of fibers of f. Its dimension is equal to $\dim(\overline{f(X_i)}) + d$.

In particular, we can conclude that X is irreducible if either of the following holds:

a) X is pure-dimensional.
b) f is closed, that is, for every W closed in X, its image $f(W)$ is closed in Y.

Proof. Let $X = X_1 \cup \ldots \cup X_r$ be the irreducible decomposition of X. For every $y \in Y$, we put $X_y := f^{-1}(y)$, and $(X_j)_y = X_y \cap X_j$. Since $X_y = \bigcup_{j=1}^r (X_j)_y$, and since X_y is irreducible, it follows that for every y, there is j such that $X_y = (X_j)_y$.

For every i, let $U_i := X_i \smallsetminus \cup_{j \neq i} X_j$. This is a nonempty open subset of X_i. Note that if $y \in f(U_i)$, then X_y can't be contained in $(X_j)_y$ for any $j \neq i$. It follows that

$$X_y = (X_i)_y \text{ for all } y \in f(U_i). \tag{1}$$

Note that some X_ℓ has to dominate Y: since f is surjective, we have $Y = \bigcup_j \overline{f(X_j)}$, and since Y is irreducible, we see that there is ℓ such that $Y = \overline{f(X_\ell)}$. In this case we also have $Y = \overline{f(U_\ell)}$, and the theorem proved in class implies that there is an open subset V of Y contained in $f(U_\ell)$. We deduce from (1) that $X_y = (X_\ell)_y$ for every $y \in V$, hence for all $j \neq \ell$, we have $X_j \smallsetminus X_\ell \subseteq f^{-1}(Y \smallsetminus V)$. Therefore $X_j = \overline{X_j \smallsetminus X_\ell}$ is contained in $f^{-1}(Y \smallsetminus V)$ (which is closed). We conclude that X_j does not dominate Y for any $j \neq \ell$.

For every i, we know by the results proved in class that

(2) $\dim(X_i)_y \geq \dim(X_i) - \dim(\overline{f(X_i)})$ for every $y \in f(X_i)$.
(3) There is an open subset W_i in $\overline{f(X_i)}$ such that for all $y \in W_i$ we have $\dim(X_i)_y = \dim(X_i) - \dim(\overline{f(X_i)})$.

Since $W_i \cap f(U_i) \neq \emptyset$, it follows from (3) and (1) that $d = \dim(X_i) - \dim(\overline{f(X_i)})$ for every i. Furthermore, for every $y \in f(X_i)$, we know by (2) that $(X_i)_y$ is a closed subset of dimension d of the irreducible variety X_y of dimension d. Therefore $X_y = (X_i)_y$ for all $y \in f(X_i)$, which says precisely that each X_i is a union of fibers of f.

In particular, the above discussion shows that if $i \neq \ell$, then $\overline{f(X_i)}$ is a proper subset of Y, and

$$\dim(X_i) = d + \dim(\overline{f(X_i)}) < d + \dim(Y) = \dim(X_\ell).$$

If X is pure-dimensional, then it follows that X is irreducible.

Suppose now that f is a closed map. Since $f(X_\ell)$ is closed, it follows that $f(X_\ell) = Y$. We have seen that X_ℓ is a union of fibers of f, hence $X_\ell = X$. This shows that X is irreducible also in this case. $\square$