Math 420

Homework Set 9

This set of problems is just for practice, not to be submitted.

We consider a finite-dimensional vector space V over $F = \mathbf{R}$ or $\mathbf{C}$, with an inner-product.

Problem 1. If $T \in \mathcal{L}(V)$ is a normal operator, show that

$$\text{null}(T) = \text{null}(T^*) \quad \text{and} \quad \text{range}(T) = \text{range}(T^*).$$

Problem 2. Let $P \in \mathcal{L}(V)$ such that $P^2 = P$. Show that there is a linear subspace U of V such that $P = P_U$ if and only if P is self-adjoint.

Problem 3. Suppose $F = \mathbf{R}$ and $T \in \mathcal{L}(V)$. Show that T is self-adjoint if and only if all pais of eigenvectors corresponding to distinct eigenvalues of T are orthogonal and

$$V = E(\lambda_1, T) \oplus \ldots \oplus E(\lambda_m, T),$$

where $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T.

Problem 4. Suppose that $T \in \mathcal{L}(V)$ is self-adjoint, $\lambda \in F$, and $\epsilon > 0$. Show that if there is $v \in V$ such that $\| v \| = 1$ and $\| Tv - \lambda v \| < \epsilon$, then there is an eigenvalue λ' of T such that $|\lambda' - \lambda| < \epsilon$.

Problem 5. Given $T \in \mathcal{L}(V)$, define $\langle u, v \rangle_T$ by

$$\langle u, v \rangle_T = \langle Tu, v \rangle.$$

Show that $\langle \cdot, \cdot \rangle_T$ is an inner product on T if and only if T is an invertible positive operator (with respect to the original inner product on V).

Problem 6. Suppose that $T \in \mathcal{L}(V)$ and s is a singular value of T. Show that there is a vector $v \in V$ such that $\| v \| = 1$ and $\| Tv \| = s$.

Problem 7. Show that if $T \in \mathcal{L}(V)$ is self-adjoint, then the singular values of T equal the absolute values of the eigenvalues of T, repeated appropriately.

Problem 8. Show that $T \in \mathcal{L}(V)$ is an isometry if and only if all the singular values of T are equal to 1.