Math 420

Solutions to the problems on Homework Set 8

Problem 1. Let V be a vector space over F (where $F = \mathbf{R}$ or $F = \mathbf{C}$), with an inner product. Show that for $u, v \in V$, we have $\langle u, v \rangle = 0$ if and only if

$$\| u \| \leq \| u + av \| \quad \text{for all} \quad a \in F.$$

Solution. Since the norm of a vector is nonnegative, we have

$$\| u \| \leq \| u + av \| \quad \text{if and only if} \quad \| u \|^2 \leq \| u + av \|^2 .$$

Since

$$\| u + av \|^2 = \langle u + av, u + av \rangle = \| u \|^2 + a\langle v, u \rangle + \overline{a}\langle u, v \rangle + |a|^2 \| v \|^2,$$

we see that $\| u \| \leq \| u + av \|$ for all $a \in F$ if and only if

$$(1) \qquad 2\mathrm{Re}\big(a\langle v, u \rangle\big) + |a|^2 \| v \|^2 \geq 0 \quad \text{for all} \quad a \in F.$$

This clearly holds if $\langle u, v \rangle = 0$ since $|a|^2 \geq 0$ and $\| v \|^2 \geq 0$. Conversely, if (1) holds for all $a \in F$, but $\langle u, v \rangle \neq 0$, let's take $a = \frac{t}{\langle v, u \rangle}$, with $t \in \mathbf{R}$. It follows from (1) that

$$2t + t^2 \frac{\| v \|^2}{|\langle v, u \rangle|^2} \quad \text{for all} \quad t \in \mathbf{R}.$$

This is clearly false, since the discriminant of this polynomial in t is $4 > 0$. We thus conclude that $\langle u, v \rangle = 0$.

Problem 2. On the real vector space $\mathcal{P}_2(\mathbf{R})$ of polynomials with coefficients in $\mathbf{R}$, of degree at most 2, consider the inner product

$$\langle p, q \rangle = \int_0^1 p(x)q(x)dx.$$

Apply the Gram-Schmidt algorithm to the basis $1, x, x^2$ to produce an orthonormal basis of $\mathcal{P}_2(\mathbf{R})$.

This is a straightforward (though somewhat messy) application of the Gram-Schmidt algorithm. I do not include the solution.

Problem 3. Suppose that V is a real vector space with an inner product and $v_1, \ldots, v_m$ is a linearly independent list of vectors in V. Prove that there exist exactly 2^m orthonormal lists $e_1, \ldots, e_m$ of vectors in V such that

$$\mathrm{span}(v_1, \ldots, v_j) = \mathrm{span}(e_1, \ldots, e_j) \quad \text{for all} \quad j \in \{1, \ldots, m\}.$$

Solution. It is enough to show that for every r, with $1 \leq r \leq m$, if we have chosen orthonormal $e_1, \ldots, e_{r-1}$ such that

$$(2) \qquad \mathrm{span}(v_1, \ldots, v_j) = \mathrm{span}(e_1, \ldots, e_j) \quad \text{for all} \quad j \in \{1, \ldots, r-1\},$$

then we have precisely 2 choices for e_r such that $e_1, \dots, e_r$ is an orthonormal set and

$$\text{span}(v_1, \dots, v_r) = \text{span}(e_1, \dots, e_r). \tag{3}$$

Since (2) holds, we have (3) if and only if we can write

$$e_r = \sum_{i=1}^{r-1} a_i e_i + b v_r$$

for some $a_1, \dots, a_{r-1}, b \in \mathbf{R}$, with $b \neq 0$. Since $e_1, \dots, e_{n-1}$ is an orthonormal set, in order for $e_1, \dots, e_r$ to be orthonormal, we also need

$$\langle e_r, e_j \rangle = 0 \quad \text{for} \quad 1 \leq j \leq r-1 \tag{4}$$

and $\| e_r \| = 1$. Condition (4) says that $a_j + b\langle v_n, e_j \rangle = 0 \quad \text{for} \quad 1 \leq j \leq r-1$. In other words, if $u = v_r - \sum_{i=1}^{r-1} \langle v_n, e_j \rangle e_i$, then $e_r = bu$. Note that $u \neq 0$, since $v_1, \dots, v_r$ are linearly independent (and thus $e_1, \dots, e_{r-1}, v_r$ are linearly independent); therefore $\| u \| \neq 0$. The condition $\| e_r \| = 1$ is equivalent to $|b| = \frac{1}{\|u\|}$. Therefore the only possibilities for b are $b = \pm \frac{1}{\|u\|}$. Therefore we have precisely two choices for e_r.

Problem 4. Let V be a finite dimensional vector space with an inner product. Show that if U and W are linear subspaces of V, then $P_U P_W = 0$ if and only if $\langle u, w \rangle = 0$ for every $u \in U$ and every $w \in W$.

Solution. Note that for $v \in V$, we have $P_U(v) = 0$ if and only if $v \in U^\perp$. Therefore $P_U P_W = 0$ if and only if $P_W(v) \in U^\perp$ for every $v \in V$. Also, note that we have $\langle u, w \rangle = 0$ for every $u \in U$ and every $w \in W$ if and only if $W \subseteq U^\perp$.

It is now clear that if $P_W(v) \in U^\perp$ for every $v \in V$, then for every $w \in W$, we have $w = P_W(w) \in U^\perp$. Therefore $W \subseteq U^\perp$.

Conversely, if $W \subseteq U^\perp$, then for every $v \in V$, we have $P_W(v) \in W \subseteq U^\perp$. This completes the proof.

Problem 5. Let V be a finite-dimensional inner product vector space and let $T \in \mathcal{L}(V)$. Show that if U is a linear subspace of V, then both U and $U^\perp$ are invariant under T if and only if $P_U T = T P_U$.

Solution. Suppose first that both U and $U^\perp$ are invariant under T. Given $v \in V$, if we write it as $v = v_1 + v_2$, with $v_1 \in U$ and $v_2 \in U^\perp$, then $P_U(v) = v_1$ and thus $TP_U(v) = T(v_1)$. On the other hand, we have $T(v) = T(v_1) + T(v_2)$, with $T(v_1) \in U$ and $T(v_2) \in U^\perp$ (we use here that U and $U^\perp$ are invariant under T). Therefore $P_U T(v) = T(v_1) = TP_U(v)$.

Conversely, suppose that $P_U T = T P_U$. Note that

$$U = \{v \in V \mid P_U(v) = v\} \quad \text{and} \quad U^\perp = \{v \in V \mid P_U(v) = 0\}.$$

If $v \in U$, we thus have $P_U T(v) = T P_U(v) = T(v)$, hence $T(v) \in U$. Similarly, if $v \in U^\perp$, then $P_U T(v) = T P_U(v) = T(0) = 0$, hence $T(v) \in U^\perp$. This completes the proof.

Problem 6. In $\mathbf{R}^4$, let

$$U = \operatorname{span}\big((1,1,0,0),(1,1,1,2)\big).$$

Find $u \in U$ such that the distance between u and $(1,2,3,4)$ is as small as possible.

Solution. By the proposition we proved in class, we need to find the orthogonal projection of $(1,2,3,4)$ onto U. In other words, we need to find $u = a(1,1,0,0) + b(1,1,1,2)$ such that

$$\langle (1,2,3,4) - u, (1,1,0,0)\rangle = 0 \quad \text{and} \quad \langle (1,2,3,4) - u, (1,1,1,2)\rangle = 0.$$

The two conditions are

$$3 - 2a - 2b = 0 \quad \text{and} \quad 14 - 2a - 7b = 0,$$

We obtain $a = -\frac{7}{10}$ and $b = \frac{11}{5}$, hence

$$u = \left(\frac{3}{2}, \frac{3}{2}, \frac{11}{5}, \frac{22}{5}\right).$$