Math 420

Homework Set 8

This assignment is due on Monday, December 9.

Problem 1. Let V be a vector space over F (where $F = \mathbf{R}$ or $F = \mathbf{C}$), with an inner product. Show that for $u, v \in V$, we have $\langle u, v \rangle = 0$ if and only if

$$\| u \| \leq \| u + av \| \quad \text{for all} \quad a \in F.$$

Problem 2. On the real vector space $\mathcal{P}_2(\mathbf{R})$ of polynomials with coefficients in $\mathbf{R}$, of degree at most 2, consider the inner product

$$\langle p, q \rangle = \int_0^1 p(x)q(x)dx.$$

Apply the Gram-Schmidt algorithm to the basis $1, x, x^2$ to produce an orthonormal basis of $\mathcal{P}_2(\mathbf{R})$.

Problem 3. Suppose that V is a real vector space with an inner product and $v_1, \ldots, v_m$ is a linearly independent list of vectors in V. Prove that there exist exactly 2^m orthonormal lists $e_1, \ldots, e_m$ of vectors in V such that

$$\text{span}(v_1, \ldots, v_j) = \text{span}(e_1, \ldots, e_j) \quad \text{for all} \quad j \in \{1, \ldots, m\}.$$

Problem 4. Let V be a finite dimensional vector space with an inner product. Show that if U and W are linear subspaces of V, then $P_U P_W = 0$ if and only if $\langle u, w \rangle = 0$ for every $u \in U$ and every $w \in W$.

Problem 5. Let V be a finite-dimensional inner product vector space and let $T \in \mathcal{L}(V)$. Show that if U is a linear subspace of V, then both U and $U^\perp$ are invariant under T if and only if $P_U T = T P_U$.

Problem 6. In $\mathbf{R}^4$, let

$$U = \text{span}\big((1, 1, 0, 0), (1, 1, 1, 2)\big).$$

Find $u \in U$ such that the distance between u and $(1, 2, 3, 4)$ is as small as possible.