Homework Set 7

Solutions are due Thursday, March 8.

Problem 1. Show that if X is an algebraic variety with irreducible components $X_1, \ldots, X_r$, then X is affine if and only if X_i is affine for $1 \leq i \leq r$.

Problem 2. Given a short exact sequence of $\mathcal{O}_X$-modules on the algebraic variety X:

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

show that if $\mathcal{F}'$ and $\mathcal{F}''$ are quasi-coherent (coherent), then $\mathcal{F}$ is quasi-coherent (respectively, coherent).

Problem 3. Show that if $\mathcal{F}$ is a sheaf (say, of Abelian groups) on X and there is an open cover $X = \bigcup_{i \in I} U_i$ such that $\mathcal{F}|_{U_i}$ is flasque for every i, then $\mathcal{F}$ is flasque.

Problem 4. Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Show that if $\mathcal{E}$ is a locally free sheaf on Y, then for every $\mathcal{O}_X$-module $\mathcal{F}$ on X, we have an isomorphism

$$R^p f_*\big(f^*(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{F}\big) \simeq \mathcal{E} \otimes_{\mathcal{O}_Y} R^p f_*(\mathcal{F}) \quad \text{for all} \quad p \geq 0.$$

Problem 5. Show that if X is an irreducible variety, then we have an isomorphism

$$\mathrm{Pic}(X) \simeq H^1(X, \mathcal{O}_X^*).$$