Math 420

Homework Set 7

This assignment is due on Wednesday, November 27.

Problem 1. Let V be a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$ such that $T^2 = T$. Show that

$$\operatorname{trace}(T) = \dim\big(\operatorname{range}(T)\big).$$

Problem 2.

i) Let V be a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Show that $\det(T) = \det(T^*)$, where T^* is the corresponding operator on the dual vector space V^*. Hint: choose a basis of V such that T is given by an upper-triangular matrix.

ii) Show that if $A \in M_n(\mathbf{R})$ and A^* is the transpose of A, then we also have $\det(A) = \det(A^*)$.

Problem 3. A matrix $A = (a_{i,j}) \in M_n(\mathbf{C})$ is *skew-symmetric* if $a_{i,j} = -a_{j,i}$ for all i and j. Show that if A is skew-symmetric and n is *odd*, then $\det(A) = 0$.

Problem 4. Let V be a finite-dimensional complex vector space and suppose that S and T are diagonalizable operators on V. Show that if $ST = TS$, then there is a basis of V such that, with respect to this basis, both S and T are given by diagonal matrices.

Problem 5. Let $A = (a_{i,j}) \in M_n(F)$ and let $B = (b_{i,j}) \in M_n(F)$ be the matrix such that

$$b_{i,j} = (-1)^{i+j} a_{i,j} \quad \text{for} \quad 1 \leq i, j \leq n.$$

Show that

$$\det(B) = \det(A).$$