Homework Set 6

Solutions are due Thursday, February 22.

Problem 1. Show that if $(X, \mathcal{O}_X)$ is a ringed space and

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is a morphism of $\mathcal{O}_X$-modules, with $\mathcal{F}'$ flasque, then we have a short exact sequence

$$0 \to \mathcal{F}'(X) \to \mathcal{F}(X) \to \mathcal{F}''(X) \to 0.$$

Hint: given $s'' \in \mathcal{F}''(X)$, consider a lift $s \in \mathcal{F}(U)$ of $s''|_U$, with U maximal, and argue that $U = X$.

Problem 2. Let $(X, \mathcal{O}_X)$ be a ringed space.

i) Given an open subset U of X, with $i\colon U \hookrightarrow X$ the inclusion map, define the ideal $i_!(\mathcal{O}_U)$ of $\mathcal{O}_X$ by

$$\Gamma\big(V, i_!(\mathcal{O}_U)\big) = \Gamma\big(V, \mathcal{O}_U(V)\big) \quad \text{if} \quad V \subseteq U,$$

and $\Gamma\big(V, i_!(\mathcal{O}_U)\big) = 0$ if $V \not\subseteq U$. Show that for every $\mathcal{O}_X$-module $\mathcal{F}$ we have a natural isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}\big(i_!(\mathcal{O}_U), \mathcal{F}\big) \simeq \mathcal{F}(U).$$

ii) Deduce that every injective $\mathcal{O}_X$-module is flasque.

The next problem concerns the étale space of a presheaf. Its main purpose is to describe each sheaf as the sheaf of sections of some continuous map. Moreover, it is quite useful for dealing with sections of a sheaf over non-empty subsets.

If $\mathcal{F}$ is a sheaf on a topological space X and Z is an arbitrary subset of X, with $i\colon Z \hookrightarrow X$ the inclusion map, then we put

$$\mathcal{F}(Z) = \Gamma(Z, \mathcal{F}) := \Gamma\big(Z, i^{-1}(\mathcal{F})\big).$$

Problem 3. Let X be a topological space and $\mathcal{F}$ a presheaf on X (say, of Abelian groups). We let $\mathrm{Et}(\mathcal{F}) = \bigsqcup_{x \in X} \mathcal{F}_x$ and consider the map $\pi\colon \mathrm{Et}(\mathcal{F}) \to X$ that maps the stalk $\mathcal{F}_x$ to $x \in X$. For every open subset U of X and every $s \in \mathcal{F}(U)$, consider the map $\widetilde{s}\colon U \to \mathrm{Et}(\mathcal{F})$ given by $s(x) = s_x \in \mathcal{F}_x$ (we thus have $\pi \circ \widetilde{s}(x) = x$ for all $x \in U$). We consider on $\mathrm{Et}(\mathcal{F})$ the strongest topology that makes all maps $\widetilde{s}$ continuous; explicitly, a subset $V \subseteq \mathrm{Et}(\mathcal{F})$ is open if and only if for every map $\widetilde{s}$ as above, the subset $\widetilde{s}^{-1}(V)$ of U is open.

i) Show that for every open subset U of X and every $s \in \mathcal{F}(U)$, the subset $\widetilde{s}(U)$ of $\mathrm{Et}(\mathcal{F})$ is open.
ii) Show that π is a local homeomorphism of $\mathrm{Et}(\mathcal{F})$ onto X.

iii) Show that the subsets $\widetilde{s}(U)$, with U open in X and $s \in \mathcal{F}(U)$, give a basis for the topology of $\mathrm{Et}(\mathcal{F})$.

iv) Show that if $\mathcal{F}^+$ is the sheaf associated to $\mathcal{F}$, then $\mathcal{F}^+$ is naturally isomorphic to the sheaf of sections of π (this is the sheaf that maps U to the set of continuous maps $\phi\colon U \to \mathrm{Et}(\mathcal{F})$ such that $\pi \circ \phi(x) = x$ for all $x \in U$).

Suppose now that $\mathcal{F}$ is a sheaf.

v) Given a subset Z of X (with the induced topology), show that we can identify $\Gamma(Z, \mathcal{F})$ with the set of continuous maps $f\colon Z \to \mathrm{Et}(\mathcal{F})$ such that $\pi \circ f(x) = x$ for all $x \in Z$.

vi) Deduce that if $X = \bigcup_{i \in I} Z_i$ is a locally finite[1] cover of X by closed subsets, then the following sequence induced by the restriction maps

$$0 \to \mathcal{F}(X) \to \prod_{i \in I} \mathcal{F}(Z_i) \to \prod_{i,j \in I} \mathcal{F}(Z_i \cap Z_j)$$

is exact.

[1]A cover $X = \bigcup_{j \in J} V_j$ is *locally finite* if every $x \in X$ has an open neighborhood V that intersects only finitely many of the V_j.