Homework Set 6

Solutions are due Wednesday, November 1.

All presheaves considered below are either of R-modules or of R-algebras, for some commutative ring R.

Problem 1. Show that if $\phi\colon \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then the following are equivalent:

i) The morphism ϕ is an isomorphism.
ii) There is an open cover $X = \bigcup_i U_i$ such that $\phi|_{U_i}$ is an isomorphism for all i.
iii) For every $x \in X$, the induced morphism ϕ_x is an isomorphism.

Problem 2. Let $\mathcal{F}$ be a sheaf and $\mathcal{F}_1$ and $\mathcal{F}_2$ be subsheaves of $\mathcal{F}$.

i) Show that if there is an open cover $X = \bigcup_{i \in I} U_i$ such that $\mathcal{F}_1|_{U_i} \subseteq \mathcal{F}_2|_{U_i}$ for every i, then $\mathcal{F}_1 \subseteq \mathcal{F}_2$.
ii) Show that if $\mathcal{F}_{1,x} \subseteq \mathcal{F}_{2,x}$ for every $x \in X$, then $\mathcal{F}_1 \subseteq \mathcal{F}_2$.

Problem 3. For every prevarieties X and Y, with X affine, show that the map

$$\mathrm{Hom}(Y, X) \to \mathrm{Hom}_{k-\mathrm{alg}}\big(\mathcal{O}_X(X), \mathcal{O}_Y(Y)\big)$$

is a bijection.

Problem 4. Let X be a prevariety and $f \in \Gamma(X, \mathcal{O}_X)$. Recall that

$$D_X(f) = \{x \in X \mid f(x) \neq 0\}.$$

i) Show that the restriction map

$$\Gamma(X, \mathcal{O}_X) \to \Gamma\big(D_X(f), \mathcal{O}_X\big)$$

induces a ring homomorphism

$$\Gamma(X, \mathcal{O}_X)_f \to \Gamma(D_X(f), \mathcal{O}_X). \tag{1}$$

ii) Show that the morphism in (1) is an isomorphism.

The following is an extra credit problem.

Problem 5. Let X be a prevariety and let $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$ such that the ideal they generate is $\Gamma(X, \mathcal{O}_X)$. Show that if $D_X(f_i)$ is an affine variety for every i, then X is an affine variety.