Math 420

Solutions for Homework Set 6

Problem 1. Show that if V is a vector space over F, with $\dim(V) = n$, and if $T \in \mathcal{L}(V)$, then for every $j \geq n$, we have

$$V = \text{null}(T^j) \oplus \text{range}(T^j).$$

Solution. By one of the problems on the previous problem set, for $S \in \mathcal{L}(V)$, we have $V = \text{null}(S) \oplus \text{range}(S)$ if and only if $\text{null}(S) \cap \text{range}(S) = \{0\}$. Therefore we only need to show that

$$\text{null}(T^j) \cap \text{range}(T^j) = \{0\}.$$

Suppose that $u \in \text{null}(T^j) \cap \text{range}(T^)$. Therefore $T^j(u) = 0$ and there is $v \in V$ such that $u = T^j(v)$. In particular, we have $T^{2j}(v) = 0$. On the other hand, we have shown in class that

$$\text{null}(T^m) = \text{null}(T^{m+1})$$

for every $m \geq n$. Since $j \geq n$, we thus have $\text{null}(T^j) = \text{null}(T^{2j})$. We have seen that $v \in \text{null}(T^{2j})$, hence $v \in \text{null}(T^j)$, which implies that $u = T^j(v) = 0$. This shows that $\text{null}(T^j) \cap \text{range}(T^j) = \{0\}$.

Problem 2. Let V be a vector space over F and let $T \in \mathcal{L}(V)$.

i) Show that if T is nilpotent and λ is an eigenvalue of T, then $\lambda = 0$.
ii) Show that if $F = \mathbf{C}$ and the only eigenvalue of T is 0, then T is nilpotent.

Solution. Suppose first that T is nilpotent, hence $T^N = 0$ for some $N \geq 1$. If λ is an eigenvalue of T, there is a nonzero $v \in V$ such that $Tv = \lambda v$. This easily implies that $T^m v = \lambda^m v$ for every $m \geq 1$ (in fact, we have discussed this in class). By taking $m = N$, we have $0 = T^N v = \lambda^m v$. Since $v \neq 0$, it follows that $\lambda^m = 0$, hence $\lambda = 0$.

Suppose now that we work over $\mathbf{C}$ and the only eigenvalue of T is 0. In this case, the characteristic polynomial of T is $p_T(x) = x^n$, where $n = \dim(V)$. The Cayley-Hamilton theorem thus implies $T^n = 0$, hence T is nilpotent.

Problem 3. Let V be a finite-dimensional vector space over $\mathbf{C}$, $T \in \mathcal{L}(V)$, and

$$q(x) = x^r + a_{r-1}x^{r-1} + \ldots + a_1 x + a_0$$

the minimal polynomial of T. Show that T is invertible if and only if $a_0 \neq 0$.

Solution. Suppose first that $a_0 \neq 0$. It follows from the definition of the minimal polynomial that we have

$$T^r + a_{r-1}T^{r-1} + \ldots + a_1 T + a_0 I = 0.$$

By multiplying with $-1/a_0$ and taking out a factor of T from all but the last one of the terms above, we obtain

$$T \circ \big((-1/a_0)T^{r-1} + (-a_{r-1}/a_0)T^{r-2} + \ldots + (-a_1/a_0)I\big) = I$$
$$= \big((-1/a_0)T^{r-1} + (-a_{r-1}/a_0)T^{r-2} + \ldots + (-a_1/a_0)I\big) \circ T.$$

Therefore T is invertible.

Conversely, suppose that T is invertible. Arguing by contradiction, let us assume that $a_0 = 0$. By definition of the minimal polynomial, we have

$$T^r + a_{r-1}T^{r-1} + \ldots + a_1 T = 0.$$

Thus implies that for every $u \in V$, we have

$$T(T^{r-1}u + a_{r-1}T^{r-2}u + \ldots + a_1 u) = 0.$$

Since T is injective, we have $\mathrm{null}(T) = \{0\}$, hence

$$T^{r-1}u + a_{r-1}T^{r-2}u + \ldots + a_1 u = 0.$$

Since this holds for every $u \in V$, we conclude that

$$T^{r-1} + a_{r-1}T^{r-2} + \ldots + a_1 I = 0.$$

We thus found a monic polynomial of degree smaller than r that vanishes on T, contradicting the fact that $q(x)$ was the minimal polynomial of T.

Problem 4. Give an example of an operator on $\mathbf{C}^4$ whose characteristic polynomial equals $(x-1)(x-5)^3$ and whose minimal polynomial equals $(x-1)(x-5)^2$.

Solution. Consider the operator T given in the standard basis of $\mathbf{C}^4$ by the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Since A is in Jordan canonical form, the eigenvalues of T are 1 (with multiplicity 1) and 5 (with multiplicity 3). Therefore the characteristic polynomial of T is $p(x) = (x-1)(x-5)^3$. Since the minimal polynomial $q(x)$ of T is a monic polynomial that divides $p(x)$ and has the same roots, it follows that $q(x)$ is the polynomial of smallest degree that satisfies $q(T) = 0$ out of the following polynomials $(x-1)(x-5)$, $(x-1)(x-5)^2$, and $(x-1)(x-5)^3$.

By direct computation, we see that

$$(A-I)(A-5I) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$(A-I)(A-5I)^2 = 0,$$

hence the minimal polynomial of T is indeed $(x-1)(x-5)^2$.

Problem 5. Let V be a finite-dimensional vector space over $\mathbf{C}$. Show that for every $T \in \mathcal{L}(V)$, there are operators T_s and T_n in $\mathcal{L}(V)$, such that the following conditions holds:

i) $T = T_s + T_n$.
ii) T_s and T_n commute (that is, we have $T_s T_n = T_n T_s$).
iii) T_s is diagonalizable (the s-subscript stands for "semisimple", which is another name for "diagonalizable").
iv) T_n is nilpotent.

Remark. One can show that given T, the operators T_s and T_n that satisfy properties i)-iv) above are unique.

Solution. Let $\lambda_1, \ldots, \lambda_r$ be the distinct eigenvalues of T and consider the decomposition $V = G(\lambda_1, T) \oplus \ldots \oplus G(\lambda_r, T)$. We define the operator T_s such that if $u = u_1 + \ldots + u_r$, with $u_i \in G(\lambda_i, T)$, then $T_s(u) = \lambda_1 u_1 + \ldots + \lambda_r u_r$. It is straightforward to see that T_s is a diagonalizable operator (if we choose a basis of V by putting together bases for each of $G(\lambda_1, T), \ldots, G(\lambda_r, T)$, the matrix of T_s in this basis is diagonal).

We put $T_n = T - T_s$, so that condition i) follows by definition and condition iii) holds by the above discussion. Note that by definition, each $G(\lambda_i, T)$ is invariant under T_s. Since it is also invariant under T, it is also invariant under $T_n = T - T_s$. Moreover, we have $T_n|_{G(\lambda_i,T)} = (T - \lambda_i I)|_{G(\lambda_i,T)}$, hence $T_n|_{G(\lambda_i,T)}$ is nilpotent. If $d = \dim(V)$, then $T_n^d = 0$. Indeed, for every $v \in V$, we can write $v = v_1 + \ldots + v_r$, with $v_i \in G(\lambda_i, T)$. Since $d \geq \dim\big(G(\lambda_i, T)\big)$, we have $T_n^d(v_i) = 0$ for every i, hence $T_n^d(v) = \sum_i T_n^d(v_i) = 0$. Therefore T_n is nilpotent.

We are left with proving that $T_s T_n = T_n T_s$. Given any $v \in V$, we write it as $v = v_1 + \ldots + v_r$, with $v_i \in G(\lambda_i, T)$. In order to show that $T_s T_n(v) = T_n T_s(v)$ it is enough to show that $T_s T_n(v_i) = T_n T_s(v_i)$ for all i. By the definition of T_s, we have

$$T_n T_s(v_i) = T_n(\lambda_i v_i).$$

On the other hand, since $G(\lambda_i, T)$ is invariant under T_n, we have $T_n(v_i) \in G(\lambda_i, T)$, and it follows from the definition of T_s that

$$T_s T_n(v_i) = \lambda_i T_n(v_i).$$

Since T_n is a linear map, we have $T_n(\lambda_i v_i) = \lambda_i T_n(v_i)$, completing the proof.