Math 420

Homework Set 6

This assignment is due on Wednesday, November 13.

Problem 1. Show that if V is a vector space over F, with $\dim(V) = n$, and if $T \in \mathcal{L}(V)$, then for every $j \geq n$, we have

$$V = \text{null}(T^j) \oplus \text{range}(T^j).$$

Problem 2. Let V be a vector space over F and let $T \in \mathcal{L}(V)$.

i) Show that if T is nilpotent and λ is an eigenvalue of T, then $\lambda = 0$.
ii) Show that if $F = \mathbf{C}$ and the only eigenvalue of T is 0, then T is nilpotent.

Problem 3. Let V be a finite-dimensional vector space over $\mathbf{C}$, $T \in \mathcal{L}(V)$, and

$$q(x) = x^r + a_{r-1}x^{r-1} + \ldots + a_1 x + a_0$$

the minimal polynomial of T. Show that T is invertible if and only if $a_0 \neq 0$.

Problem 4. Give an example of an operator on $\mathbf{C}^4$ whose characteristic polynomial equals $(x-1)(x-5)^3$ and whose minimal polynomial equals $(x-1)(x-5)^2$.

Problem 5. Let V be a finite-dimensional vector space over $\mathbf{C}$. Show that for every $T \in \mathcal{L}(V)$, there are operators T_s and T_n in $\mathcal{L}(V)$, such that the following conditions holds:

i) $T = T_s + T_n$.
ii) T_s and T_n commute (that is, we have $T_sT_n = T_nT_s$).
iii) T_s is diagonalizable (the s-subscript stands for "semisimple", which is another name for "diagonalizable").
iv) T_n is nilpotent.

Remark. One can show that given T, the operators T_s and T_n that satisfy properties i)-iv) above are unique.