Homework Set 5

Solutions are due Thursday, February 15.

Problem 1 Show that given two surjective, finite morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ of irreducible varieties, all of them being smooth in codimension 1, we have $(g \circ f)_* = g_* \circ f_*$ as maps $\mathrm{Cl}(X) \to \mathrm{Cl}(Z)$.

Problem 2. Show that if X is a variety that is smooth in codimension 1, then for every $n \geq 1$, the product $X \times \mathbf{A}^n$ has the same property, and by mapping V to $V \times \mathbf{A}^n$ we get an isomophism

$$\mathrm{Cl}(X) \simeq \mathrm{Cl}(X \times \mathbf{A}^n).$$

Problem 3. Let X be an irreducible variety.

i) Show that for every Cartier divisors D and E, we have an isomorphism
$$\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(E) \to \mathcal{O}_X(D+E).$$
ii) Show that if D is a principal Cartier divisor, then we have an isomorphism $\mathcal{O}_X(D) \simeq \mathcal{O}_X$.
iii) Deduce that we have a group morphism
$$\mathrm{Cart}(X)/\mathrm{PCart}(X) \to \mathrm{Pic}(X)$$
that maps the class of D to the (isomorphism class) of $\mathcal{O}_X(D)$. Show that this is injective.

The next problem gives a compatibility property between push-forward and pull-back of divisors, known as the *projection formula*.

Problem 4.

If : $X \to Y$ is a finite surjective morphism between irreducible varieties, both of them being smooth in codimension 1, then for every Cartier divisor D on Y, we have the following equality[1] in $\mathrm{Div}(Y)$:

$$f_*\big(f^*(D)\big) = \deg(f) \cdot D.$$

Problem 5. Show that if $\mathcal{L}$ is a line bundle on the irreducible, complete variety X, such that $\Gamma(X, \mathcal{L}) \neq 0$ and $\Gamma(X, \mathcal{L}^{-1}) \neq 0$, then $\mathcal{L} \simeq \mathcal{O}_X$.

[1]On the left-hand side, we apply f_* to the Weil divisor corresponding to $f^*(D)$, while on the right-hand side, we consider the Weil divisor corresponding to D.