Homework Set 5

Solutions are due Wednesday, October 25.

Problem 1. Show that if X and Y are quasi-affine varieties, then

$$\dim(X \times Y) = \dim(X) + \dim(Y).$$

Problem 2. Show that if X is an affine variety such that $\mathcal{O}(X)$ is a UFD, then for every closed subset $Y \subseteq X$, having all irreducible components of codimension 1, the ideal $I_X(Y)$ defining Y is principal.

Problem 3. Show that if X and Y are irreducible closed subsets of $\mathbf{A}^n$, then every irreducible component of $X \cap Y$ has dimension $\geq \dim(X) + \dim(Y) - n$ (Hint: describe $X \cap Y$ as the intersection of $X \times Y \subseteq \mathbf{A}^n \times \mathbf{A}^n$ with the diagonal $\Delta = \{(x, x) \mid x \in \mathbf{A}^n\}$).

Problem 4. Let X be a (quasi-affine) variety, and p a point on X. Show that $\dim_p(X) := \dim(\mathcal{O}_{X,p})$ is equal to the largest dimension of an irreducible component of X that contains p.

Problem 5. Let R be a commutative ring and consider the *spectrum* of R:

$$\mathrm{Spec}(R) := \{\mathfrak{p} \mid \mathfrak{p} \text{ prime ideal in } R\}.$$

For every ideal J in R, consider

$$V(J) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid J \subseteq \mathfrak{p}\}.$$

Show that the following hold:

i) For every ideals J_1, J_2 in R, we have
$$V(J_1) \cup V(J_2) = V(J_1 \cap J_2) = V(J_1 \cdot J_2).$$
ii) For every family $(J_\alpha)_\alpha$ of ideals in R, we have
$$\bigcap_\alpha V(J_\alpha) = V\left(\sum_\alpha J_\alpha\right).$$
iii) We have
$$V(0) = \mathrm{Spec}(R) \quad \text{and} \quad V(R) = \emptyset.$$
iv) Deduce that $\mathrm{Spec}(R)$ has a topology (the *Zariski topology*) whose closed subsets are the $V(J)$, with J an ideal in R.
v) Show that $V(J) = V(J')$ if and only if $\mathrm{rad}(J) = \mathrm{rad}(J')$.
vi) Show that the closed irreducible subsets in $\mathrm{Spec}(R)$ are those of the form $V(P)$, where P is a prime ideal in R. Deduce that
$$\dim(R) = \dim\big(\mathrm{Spec}(R)\big).$$