Math 420

Solutions for Homework Set 5

Problem 1 (10 points). Let $T \in \mathcal{L}(\mathbf{R}^2)$ be given by $T(x,y) = (-3y, x)$. Find the eigenvalues of T.

Solution. A real number λ is an eigenvalue of T if and only if there are real numbers a, b, not both 0, such that $T(a,b) = \lambda(a,b)$. In other words, we need $-3b = \lambda a$ and $a = \lambda b$. From this we see immediately that if $b = 0$, then also $a = 0$, contradiction. Therefore $b \neq 0$, and since $-3b = \lambda a = \lambda^2 b$, we conclude that $\lambda^2 = -3$. Since λ must be a real number, we conclude that T has no eigenvalues.

Problem 2 (20 points). Let V be a finite-dimensional vector space over the field $\mathbf{F}$ and let $T \in \mathcal{L}(V)$.

i) Show that we have $V = \mathrm{null}(T) \oplus \mathrm{range}(T)$ if and only if $\mathrm{null}(T) \cap \mathrm{range}(T) = \{0\}$.
ii) Show that if T is diagonalizable, then $V = \mathrm{null}(T) \oplus \mathrm{range}(T)$.
iii) Give an example to show that the converse of the assertion in ii) does not hold.
iv) Show that if $\mathbf{F} = \mathbf{C}$ and for every $\lambda \in \mathbf{C}$, we have

$$V = \mathrm{null}(T - \lambda I) \oplus \mathrm{range}(T - \lambda I),$$

then T is diagonalizable.

Solution. i) The "only if" part is clear. Suppose now that $\mathrm{null}(T) \cap \mathrm{range}(T) = \{0\}$. We know that if $n = \dim(V)$, then $\dim\big(\mathrm{null}(T)\big) + \dim\big(\mathrm{range}(T)\big) = n$. The formula for the dimension of the sum of two vector spaces thus gives

$$\dim\big(\mathrm{null}(T) + \mathrm{range}(T)\big) = \dim\big(\mathrm{null}(T)\big) + \dim\big(\mathrm{range}(T)\big) - \dim\big(\mathrm{null}(T) \cap \mathrm{range}(T)\big)$$

$$= n - 0 = n.$$

This implies that $\mathrm{null}(T) + \mathrm{range}(T) = V$. Together with the assumption that $\mathrm{null}(T) \cap \mathrm{range}(T) = \{0\}$, this implies

$$V = \mathrm{null}(T) \oplus \mathrm{range}(T).$$

ii) If V is diagonalizable, then we can write

$$V = E(\lambda_1, T) \oplus \ldots \oplus E(\lambda_r, T), \tag{1}$$

where $\lambda_1, \ldots, \lambda_r$ are the distinct eigenvalues of T. If $\lambda_i \neq 0$ for any i, then T is invertible, hence $\mathrm{range}(T) = V$ and $\mathrm{null}(T) = \{0\}$, and we clearly have $\mathrm{null}(T) \cap \mathrm{range}(T) = \{0\}$.

Suppose now that $\lambda_1 = 0$. Note that $\mathrm{null}(T) = E(0, T)$. In order to obtain the assertion in ii), it is enough to show that

$$\mathrm{range}(T) = E(\lambda_2, T) \oplus \ldots \oplus E(\lambda_r, T).$$

If $u \in E(\lambda_i, T)$ for some $i \geq 2$, then $Tu = \lambda_i u$. Since $\lambda_i \neq 0$, we have

$$u = T(\lambda_i^{-1} u) \in \operatorname{range}(T).$$

This holds for every $u \in E(\lambda_i, T)$, for every $i \geq 2$, hence

$$\operatorname{range}(T) \supseteq E(\lambda_2, T) \oplus \ldots \oplus E(\lambda_r, T).$$

On the other hand, if $w \in \operatorname{range}(T)$, then we can write $w = T(v)$ for some $v \in V$. By (1), we can write

$$v = v_1 + \ldots + v_r,$$

with $v_i \in E(\lambda_i, T)$ for all i. We thus have

$$w = T(v) = \sum_{i=2}^{r} \lambda_i v_i \in E(\lambda_2, T) \oplus \ldots \oplus E(\lambda_r, T).$$

We thus have also

$$\operatorname{range}(T) \subseteq E(\lambda_2, T) \oplus \ldots \oplus E(\lambda_r, T).$$

This completes the proof of ii).

iii) As we have already mentioned, if T is invertible, we always have $V = \operatorname{null}(T) \oplus \operatorname{range}(T)$. Therefore we can take any example of an invertible T, which is not diagonalizable. For example, T given by the matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

iv) We argue by induction on $n = \dim(V)$, the case $n = 0$ being clear. Suppose now that $n > 0$ and that we know the assertion for vector spaces of dimension $< n$. Since we work over $\mathbf{C}$, T has at least one eigenvalue λ. By assumption, we can write

$$V = E(\lambda, T) \oplus W,$$

where $W = \operatorname{range}(T - \lambda I)$. In particular, W is invariant under T.

It is enough to show that $T|_W$ is diagonalizable. Since $\dim(W) < \dim(V)$, using the induction hypothesis, we see that it is enough to show that $T|_W$ also satisfies the condition T did. More precisely, we are done if we show that for every $\mu \in \mathbf{C}$, we have

$$W = \operatorname{null}(T|_W - \mu \mathrm{I}_W) \oplus \operatorname{range}(T|_W - \mu \mathrm{I}_W).$$

By part i), it is enough to show that

$$\operatorname{null}(T|_W - \mu \mathrm{I}_W) \cap \operatorname{range}(T|_W - \mu \mathrm{I}_W) = \{0\}.$$

On the other hand, it follows immediately that

$$\operatorname{null}(T|_W - \mu \mathrm{I}_W) \subseteq \operatorname{null}(T - \mu I) \quad \text{and} \quad \operatorname{range}(T|_W - \mu \mathrm{I}_W) \subseteq \operatorname{range}(T - \mu I).$$

We thus conclude that

$$\operatorname{null}(T|_W - \mu \mathrm{I}_W) \cap \operatorname{range}(T|_W - \mu \mathrm{I}_W) \subseteq \operatorname{null}(T - \mu I) \cap \operatorname{range}(T - \mu I) = \{0\}.$$

This completes the proof of iv).

Problem 3 (10 points). Let V be a finite-dimensional vector space over $\mathbf{F}$. Show that if U is a linear subspace of V that is invariant under T, then every eigenvalue of the operator T/U on V/U is an eigenvalue of T.

Proof. Suppose that λ is an eigenvalue of T/U. This implies that there is $u \in T \smallsetminus U$ such that $T(u+U) = \lambda(u+U)$; equivalently, we have $Tu - \lambda u \in U$.

Let $U' = U + \{au \mid a \in F\}$. Since $u \notin U$, it follows that U' is a linear subspace of V strictly containing U. On the other hand, for every $v \in U$ and every $a \in F$, we have

$$S(v + au) = S(v) + aS(u) \in U$$

(note that since U is invariant under T, it is also invariant under S). Therefore S induces a linear map $U' \to U$. Since $\dim(U') > \dim(U)$, it follows that there is $w \in U'$ nonzero such that $S(w) = 0$. Hence w is an eigenvector of T with eigenvalue λ.

Problem 4 (10 points). Let V be a finite-dimensional vector space over the field $\mathbf{F}$. Show that if $\lambda_1, \ldots, \lambda_r$ denote the distinct nonzero eigenvalues of T, then

$$\dim E(\lambda_1, T) + \ldots + \dim E(\lambda_r, T) \leq \dim \operatorname{range}(T).$$

Proof. Let $W = E(\lambda_1, T) + \ldots + E(\lambda_r, T)$. We have seen in class that $W = E(\lambda_1, T) \oplus \ldots \oplus E(\lambda_r, T)$, hence

$$\dim(W) = \dim E(\lambda_1, T) + \ldots + \dim E(\lambda_r, T).$$

In order to complete the proof, it is enough to show that $W \subseteq \operatorname{range}(T)$.

For every i, if $u \in E(\lambda_i, T)$, we have $T(u) = \lambda_i u$. Since $\lambda_i \neq 0$, we have

$$u = T(\lambda_i^{-1} u) \in \operatorname{range}(T).$$

This implies that $E(\lambda_i, T) \subseteq \operatorname{range}(T)$ for every i, and thus $W \subseteq \operatorname{range}(T)$. This completes the proof.