Math 420

Homework Set 5

This assignment is due on Monday, November 4.

Problem 1 (10 points). Let $T \in \mathcal{L}(\mathbf{R}^2)$ be given by $T(x, y) = (-3y, x)$. Find the eigenvalues of T.

Problem 2 (20 points). Let V be a finite-dimensional vector space over the field $\mathbf{F}$ and let $T \in \mathcal{L}(V)$.

i) Show that we have $V = \text{null}(T) \oplus \text{range}(T)$ if and only if $\text{null}(T) \cap \text{range}(T) = \{0\}$.
ii) Show that if T is diagonalizable, then $V = \text{null}(T) \oplus \text{range}(T)$.
iii) Give an example to show that the converse of the assertion in ii) does not hold.
iv) Show that if $\mathbf{F} = \mathbf{C}$ and for every $\lambda \in \mathbf{C}$, we have

$$V = \text{null}(T - \lambda I) \oplus \text{range}(T - \lambda I),$$

then T is diagonalizable.

Problem 3 (10 points). Let V be a finite-dimensional vector space over $\mathbf{F}$. Show that if U is a linear subspace of V that is invariant under T, then every eigenvalue of the operator T/U on V/U is an eigenvalue of T.

Problem 4 (10 points). Let V be a finite-dimensional vector space over the field $\mathbf{F}$. Show that if $\lambda_1, \ldots, \lambda_r$ denote the distinct nonzero eigenvalues of T, then

$$\dim E(\lambda_1, T) + \ldots + \dim E(\lambda_r, T) \leq \dim \text{range}(T).$$