Homework Set 5

Solutions are due Monday, December 14th.

Problem 1. Let $S = k[x_0, \ldots, x_n]$ be the homogeneous coordinate ring of $\mathbf{P}^n$, and let $\mathfrak{m} = (x_0, \ldots, x_n)$ be the "irrelevant" ideal.

i) Show that two homogeneous ideals I and J in S define the same closed subscheme of $\mathbf{P}^n$ if and only if there is $N > 0$ such that $\mathfrak{m}^N \cdot I \subseteq J$ and $\mathfrak{m}^N \cdot J \subseteq I$.

ii) An ideal $I \subseteq S$ is called $\mathfrak{m}$-*saturated* if for every $u \in S$ such that $\mathfrak{m} \cdot u \subseteq I$, we have $u \in I$. Show that

$$I^{\mathrm{sat}} := \{u \in I \mid \mathfrak{m}^N \cdot u \subseteq I \text{ for some } N\}$$

is an $\mathfrak{m}$-saturated ideal, and it is the smallest such ideal containing I.

iii) Show that I^{sat} is the largest ideal that defines the same closed subscheme of $\mathbf{P}^n$ as I does. Deduce that there is a bijection between closed subschemes of $\mathbf{P}^n$ and homogeneous $\mathfrak{m}$-saturated ideals in S (you can use the fact that we will prove later, saying that every closed subscheme of $\mathbf{P}^n$ is defined by some homogeneous ideal in S).

iv) Show that every radical ideal different from $\mathfrak{m}$ is $\mathfrak{m}$-aturated. Furthermore, given a homogeneous ideal I in S, the subscheme of $\mathbf{P}^n$ defined by I is reduced if and only if I^{sat} is a radical ideal.

Problem 2. *The Veronese embedding*. Let n and d be positive integers, and let $M_0, \ldots, M_N$ be the monomials in $k[x_0, \ldots, x_n]$ of degree d, suitably ordered (hence $N = \binom{n+d}{d} - 1$).

1) Show that there is a morphism of schemes $\rho_{n,d} \colon \mathbf{P}^n \to \mathbf{P}^N$ that takes the point $(a_0, \ldots, a_n)$ to the point $(M_0(a), \ldots, M_N(a))$.
2) Consider the ring homomorphism $f_{n,d} \colon k[z_0, \ldots, z_N] \to k[x_0, \ldots, x_n]$ defined by $f_d(z_i) = M_i$. Show that $\ker(f_{n,d})$ is a homogeneous prime ideal that defines in $\mathbf{P}^N$ the image of $\rho_{n,d}$ (in particular, this image is closed).
3) Show that $\rho_{n,d}$ is a closed immersion, called the d^{th} *Veronese embedding* of $\mathbf{P}^n$.
4) Show that if Z is a hypersurface of degree d in $\mathbf{P}^n$ (this means that Z is the closed subscheme defined by a homogeneous ideal generated by a polynomial of degree d), then there is a hyperplane H in $\mathbf{P}^N$ such that for every projective variety $X \subseteq \mathbf{P}^d$, the morphism $\rho_{n,d}$ induces an isomorphism between $X \cap Z$ and $\rho_{n,d}(X) \cap H$. (In other words, the Veronese embedding allows to reduce the intersection with a hypersurface to the intersection with a hyperplane).
5) The *rational normal curve* in $\mathbf{P}^n$ is the image of the Veronese embedding $\nu_{1,d} \colon \mathbf{P}^1 \to \mathbf{P}^d$, given by $\rho_{1,d}(a, b) = (a^d, a^{d-1}b, \ldots, b^d)$. Show that the rational normal curve is defined (as a subscheme) by the 2×2-minors of the matrix

$$\begin{pmatrix} z_0 & z_1 & \ldots & z_{d-1} \\ z_1 & z_2 & \ldots & z_d \end{pmatrix}.$$

For "extra credit", you can show that the ideal generated by the 2×2-minors of the above matrix is the prime ideal corresponding to the rational normal curve.

Problem 3. Use the previous problem to show that if $f \in S = k[x_0, \ldots, x_n]$ is a homogeneous polynomial of positive degree, then the open subset $D_+(f) \subseteq \mathbf{P}^n$ is an affine variety.

Problem 4. Show that if X is a separated scheme, and U and V are open affine subsets of X, then $U \cap V$ is affine.

Problem 5. Show that if $f\colon X \to Y$ is a morphism, and Y is separated, then the induced *graph morphism* $X \to X \times Y$ induced by (id_X, f) is a closed immersion.

Problem 6. Show that if

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

is a Cartesian diagram and f is separated, then so is f'. Hint: show that there is a Cartesian diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X' \times_{Y'} X' \arrow[d] \\ X \arrow[r] & X \times_Y X \end{tikzcd}$$

where the horizontal maps are the diagonal morphisms.

Problem 7. Let $\mathcal{P}$ be a property of morphisms of schemes such that

(a) A closed immersion has $\mathcal{P}$.
(b) $\mathcal{P}$ is stable under compositions.
(c) $\mathcal{P}$ is stable under base extensions.

Show that if $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms such that $g \circ f$ has $\mathcal{P}$ and g is separated, then f has $\mathcal{P}$.

N.B. This problem applies if $\mathcal{P}$ is one of the following properties: *closed immersion*, *finite*, *affine*, *separated*, *proper*.