Math 420

Homework Set 4

This assignment is due on Wednesday, October 23.

Recall that for every finite-dimensional vector space V, we defined a linear map $\alpha_V\colon V \to (V^*)^*$, such that for every $u \in V$, the linear map $\alpha_V(u)\colon V^* \to F$ is given by $\alpha_V(u)\varphi = \varphi(u)$. We showed in class that α_V is an isomorphism.

Problem 1. Let $f\colon V \to W$ be a linear map between finite-dimensional vector spaces.

i) Show that the following diagram is commutative

$$\begin{array}{ccc} V & \xrightarrow{\ f\ } & W \\ {\scriptstyle \alpha_V}\downarrow & & \downarrow{\scriptstyle \alpha_W} \\ (V^*)^* & \xrightarrow{(f^*)^*} & (W^*)^*, \end{array}$$

that is, $\alpha_W \circ f = (f^*)^* \circ \alpha_V$.

ii) Recall that we showed in class that f^* is injective if and only if f is surjective. Use this and the assertion in part i) to show the following: given a linear map $f\colon V \to W$, f^* is surjective if and only if f is injective.

Problem 2. Let V be a finite-dimensional vector space. Show that if A_1 and A_2 are affine subspaces of V, then $A_1 \cap A_2$ is either empty or an affine subspace of V.

Problem 3. Show that if W is a vector subspace of the finite-dimensional vector space V and $e_1, \ldots, e_n$ is a basis of V such that $e_1, \ldots, e_r$ is a basis of W, then $\overline{e_{r+1}}, \ldots, \overline{e_n}$ give a basis for V/W, where $\overline{e_i} = e_i + W$.

The result in the next problem is important, showing that for every surjective linear map $V \to W$, there is an isomorphism between W and a quotient space of V.

Problem 4. Let $f\colon V \to W$ be a linear map between finite-dimensional vector spaces and let $U = \mathrm{null}(f)$.

i) Show that if we define $g\colon V/U \to W$ by $g(a+U) = f(a)$, then g is well-defined and it is a linear map.
ii) Show that g is injective and $\mathrm{range}(g) = \mathrm{range}(f)$.
iii) Deduce that $\mathrm{range}(f)$ is isomorphic to V/U.