Math 632. Homework Set 4

Solutions are due Tuesday, April 6.

All our schemes are of finite type over an algebraically closed field k.

Problem 1. Show that every automorphism $\phi\colon \mathbf{P}^n \to \mathbf{P}^n$ is linear, that is, it is induced by an element of PGL_n.

Problem 2. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two line bundles on the scheme X.

i) Show that if $\mathcal{L}_1$ is ample and $\mathcal{L}_2$ is generated by global sections, then $\mathcal{L}_1 \otimes \mathcal{L}_2$ is ample.
ii) Show that if $\mathcal{L}_1$ is ample, then for any $\mathcal{L}_2$ we have $\mathcal{L}_1^m \otimes \mathcal{L}_2$ ample if $m \gg 0$.
iii) Show that if both $\mathcal{L}_1$ and $\mathcal{L}_2$ are ample, then so is $\mathcal{L}_1 \otimes \mathcal{L}_2$.
iv) Show that if $\mathcal{L}_1$ is very ample, and $\mathcal{L}_2$ is generated by global sections, then $\mathcal{L}_1 \otimes \mathcal{L}_2$ is very ample.
v) Show that if $\mathcal{L}_1$ is ample, then $\mathcal{L}_1^{\otimes m}$ is very ample for $m \gg 0$.

Problem 3. Let $X = \mathbf{P}^m \times \mathbf{P}^n$ be a product of projective spaces, and let $p\colon X \to \mathbf{P}^m$ and $q\colon X \to \mathbf{P}^n$ be the two projections.

i) Show that every line bundle $\mathcal{L}$ on X is isomorphic to $p^*\mathcal{O}_{\mathbf{P}^m}(a) \otimes q^*\mathcal{O}_{\mathbf{P}^n}(b)$ for unique $a, b \in \mathbf{Z}$ (in this case one says that $\mathcal{L}$ has type (a, b)).
ii) Show that a line bundle of type (a, b) is ample if and only if $a, b > 0$.

Problem 4. Show that if X and Y are nonsingular complete connected curves (that is, schemes of dimension one), then X and Y are birational if and only if they are isomorphic.

Problem 5. Prove that a scheme X is affine if and only if the structure sheaf $\mathcal{O}_X$ is ample.

Problem 6.

i) Show that if $f\colon X \to Y$ is a finite morphism of complete varieties, and $\mathcal{L}$ is an ample line bundle on Y, then $f^*(\mathcal{L})$ is ample on X.
ii) Deduce that the normalization of a projective variety is projective.
iii) For extra credit, prove the converse of the assertion in i): if $f\colon X \to Y$ is a finite surjective morphism, and $\mathcal{L}$ is a line bundle on Y such that $f^*(\mathcal{L})$ is ample on X, then $\mathcal{L}$ is ample.