Homework Set 4

Solutions are due Monday, November 23rd.

As usual, all our schemes are assumed to be of finite type over an algebraically closed field k.

Problem 1.

i) Let $f\colon Y \to X$ be a closed immersion of schemes. Show that for every scheme Z, the induced map $\mathrm{Hom}(Z, Y) \to \mathrm{Hom}(Z, X)$ (where we denote by Hom the set of scheme morphisms) is injective.
ii) Show that if f is as above, and if $g\colon Z \to X$ is a morphism of schemes such that the set-theoretic image of g is contained in the set-theoretic image of f, then it does not necessarily follow that there is a morphism $h\colon Z \to Y$ such that $f \circ h = g$.

Problem 2. Given two closed immersions $i_1\colon Y_1 \to X$ and $i_2\colon Y_2 \to X$, we put $i_1 \leq i_2$ if there is a morphism of schemes $f\colon Y_1 \to Y_2$ such that $i_2 \circ f = i_1$ (note that in this case f is unique by the previous problem). This defines an order relation on the closed immersions into X.

i) Show that if f is as above, then f is a closed immersion.
ii) Show that if $i_1 \leq i_2$ and $i_2 \leq i_1$, then the above f is an isomorphism (in this case, we say the i_1 and i_2 are equivalent).
iii) Show that we can identify the closed subschemes of X with the equivalence classes with respect to this equivalence relation.

Problem 3.

i) If Y is a closed subscheme of X, and if U is open in X, then $U \cap Y$ has a natural scheme structure as open subscheme of Y. Show that with this scheme structure, $U \cap Y$ is a closed subscheme of U.
ii) A morphism $f\colon Z \to X$ is a *locally closed immersion* if it factors as $Z \xrightarrow{i} W \xrightarrow{j} Y$, where i is a closed immersion and j is an open immersion. Deduce from i) that the class of locally closed immersions is closed under composition.

Problem 4. Prove the following criterion for gluing closed subschemes. Suppose that X is a scheme, and that we have an open cover $X = \bigcup_i U_i$. Suppose that for every i we have a closed subscheme Y_i of U_i, such that for all i and j we have $Y_i \cap U_j = Y_j \cap U_i$ (as closed subschemes of $U_i \cap U_j$). Show that in this case there is a unique closed subscheme Y of X such that for every i, $Y \cap U_i = Y_i$ as closed subschemes of U_i.

Problem 5. Let X be a scheme, and $Y \subseteq X$ a closed subset.

i) Show that there is at most one closed subscheme of X that is reduced, and whose support is equal to Y.
ii) Show that there is a closed subscheme of X, denoted Y_{red}, as in i). Hint: show this first for the affine open subsets of X, and then glue the corresponding closed subschemes using the previous problem, and i).
iii) Show that given any closed subscheme Z of X whose support contains Y, we have $Y_{\mathrm{red}} \leq Z$ (in the sense of Pb. 2).
iv) If Y' is another closed subscheme of X with support Y, then we have a surjection of sheaves $\mathcal{O}_{Y'} \to \mathcal{O}_{Y_{\mathrm{red}}}$. Show that there is an isomorphism of $\mathcal{O}_{Y_{\mathrm{red}}}$ with the image of the morphism of sheaves $\phi\colon \mathcal{O}_{Y'} \to \mathcal{C}_Y$, where $\mathcal{C}_Y$ is the sheaf of continuous functions on Y with values in k, and $\phi(u) = \widetilde{u}$.

Problem 6. Show that taking X to X_{red} extends to a functor from the category of schemes to itself.

Problem 7. Let Y be a scheme and $\{Y_\alpha\}_\alpha$ a family of closed subschemes of Y. Show that there is a unique closed subscheme of Y that is contained in all Y_α and which is maximal with this property. This is usually denoted by $\cap_\alpha Y_\alpha$. Hint: do the construction locally, and use the gluing method from Problem 4).

Problem 8. Let $Y_1, \ldots, Y_n$ be closed subschemes of a scheme Y. Show that there is a unique minimal element in the set of all closed subschemes of Y that contain all the Y_i. This is denoted by $Y_1 \cup \ldots \cup Y_n$. Hint: use the same method as in the previous problem.