Homework Set 3

Solutions are due Tuesday, January 30.

Problem 1. Show that if X is an algebraic variety and $\mathcal{A}$ is a quasi-coherent, reduced, finitely generated $\mathcal{O}_X$-algebra, then for every variety over X given by $g\colon Z \to X$, we have a canonical bijection

$$\mathrm{Hom}_{\mathrm{Var}/X}\big(Z, \mathcal{M}axSpec(\mathcal{A})\big) \to \mathrm{Hom}_{\mathcal{O}_X-\mathrm{alg}}\big(g^*(\mathcal{A}), \mathcal{O}_Z\big).$$

Problem 2. Let X be an algebraic variety, $\pi\colon E \to X$ a geometric vector bundle, and $\mathcal{E}$ the corresponding sheaf of sections. Show that for every $x \in X$, we have a canonical isomorphism of k-vector spaces between $\mathcal{E}_{(x)}$ and $\pi^{-1}(x)$.

Problem 3. Let R be a commutative ring and $\phi\colon A \to B$ a morphism of commutative R-algebras.

i) Show that for every B-module M, we have a short exact sequence of B-modules

$$0 \longrightarrow \mathrm{Der}_A(B, M) \xrightarrow{u_M} \mathrm{Der}_R(B, M) \xrightarrow{v_M} \mathrm{Der}_R(A, M),$$

where $u_M(D) = D$ and $v_M(D) = D \circ \phi$.

ii) Show that a sequence of B-modules

$$N' \to N \to N'' \to 0$$

is exact if and only if for every B-module M, the induced sequence

$$0 \to \mathrm{Hom}_B(N'', M) \to \mathrm{Hom}_B(N, M) \to \mathrm{Hom}_B(N', M)$$

is exact.

iii) Show that under our assumptions, there are morphisms of B-modules

$$\alpha\colon \Omega_{A/R} \otimes_A B \to \Omega_{B/R} \quad \text{and} \quad \beta\colon \Omega_{B/R} \to \Omega_{B/A}$$

given by

$$\alpha(d_{A/R}(a) \otimes b) = b \cdot d_{B/R}\big(\phi(a)\big) \quad \text{and} \quad \beta\big(d_{B/R}(b)\big) = d_{B/A}(b)$$

such that the sequence

$$\Omega_{A/R} \otimes_A B \xrightarrow{\alpha} \Omega_{B/R} \xrightarrow{\beta} \Omega_{B/A} \longrightarrow 0$$

is exact.

Problem 4. Let R be a commutative ring and $\phi\colon A \to B$ a *surjective* morphism of commutative R-algebras, with $I = \ker(\phi)$.

i) Show that for every B-module M, there is an exact sequence of B-modules

$$0 \longrightarrow \mathrm{Der}_R(B, M) \xrightarrow{v_M} \mathrm{Der}_R(A, M) \xrightarrow{w_M} \mathrm{Hom}_B(I/I^2, M),$$

where v_M is the same as in the previous problem and $w_M(D)$ maps $\overline{a}$ to $D(a) \in M$ for every $a \in I$.

ii) Deduce that we have a morphism of B-modules $\delta\colon I/I^2 \to \Omega_{A/R} \otimes_A B$ given by $\delta(\overline{a}) = d_{A/R}(a) \otimes 1$ for every $a \in I$ that fits in an exact sequence

$$I/I^2 \xrightarrow{\delta} \Omega_{A/R} \otimes_A B \xrightarrow{\alpha} \Omega_{B/R} \longrightarrow 0,$$

where α is the morphism in the previous problem.