Math 420

Solutions for Homework Set 3

The first problem concerns the *product of finitely many vector spaces*. Let $V_1, \ldots, V_r$ be vector spaces over the field F. Recall that the Cartesian product $\prod_{i=1}^r V_i$ (also written $V_1 \times \ldots \times V_r$) consists of all n-tuples $(u_1, \ldots, u_r)$, where $u_i \in V_i$ for $1 \leq i \leq r$. On $\prod_{i=1}^r V_i$ we define addition and scalar multiplication by

$$(u_1, \ldots, u_r) + (v_1, \ldots, v_r) := (u_1 + v_1, \ldots, u_r + v_r)$$

$$\lambda(u_1, \ldots, u_r) = (\lambda u_1, \ldots, \lambda u_r).$$

It is easy to check that with these operations $\prod_{i=1}^r V_i$ is a vector space; this is called the *product* of the vector spaces $V_1, \ldots, V_r$. Note that if $V_i = F$ for all i, then we recover the vector space F^r.

Problem 1. Let $V_1, \ldots, V_r$ be vector spaces over F as above and let $V = \prod_{i=1}^r V_i$.

i) Show that if

$$W_i = \{u = (u_1, \ldots, u_r) \in V \mid u_j = 0 \text{ for all } j \neq i\},$$

then W_i is a linear subspace of V and that W_i is isomorphic to V_i.

ii) Show that $V = W_1 \oplus \ldots \oplus W_r$.

iii) Show that if W is any vector space and $W_1, \ldots, W_r$ are linear subspaces such that $W = W_1 \oplus \ldots \oplus W_r$, then W is isomorphic to $\prod_{i=1}^r W_i$.

Solution. i) It is clear that $0 \in W_i$, hence in order to show that W_i is a linear subspace, we only need to show that it is closed under addition and scalar multiplication. Given $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n) \in W_i$, then

$$u + v = (u_1 + v_1, \ldots, u_n + v_n)$$

and for every $j \neq i$, we have $u_j + v_j = 0 + 0 = 0$. Similarly, if $u = (u_1, \ldots, u_n) \in W_i$ and $a \in F$, then

$$au = (au_1, \ldots, au_n)$$

and for $j \neq i$, we have $au_j = a0 = 0$. Therefore W_i is a linear subspace of V.

Consider now the map $f_i \colon V_i \to W_i$ given by $f_i(t) = (0, \ldots, t, \ldots, 0)$, where t appears as the i^{th} entry. It is straightforward to see that f_i is a linear map and that it is injective and bijective. This gives the isomorphism $V_i \simeq W_i$.

ii) We need to show that given $u = (u_1, \ldots, u_r) \in V$, there are unique elements $v^{(j)} \in W_j$, for $1 \leq j \leq r$ such that $u = v^{(1)} + \ldots + v^{(r)}$. It is clear that we can take $v^{(j)} = (0, \ldots, u_j, \ldots, 0)$, with u_j appearing on the j^{th} spot, and that this is, in fact, the only possibility.

iii) Define the map $f \colon \prod_{i=1}^r W_i \to W$ by

$$f(w_1, \ldots, w_r) = w_1 + \ldots + w_r.$$

It is straightforward to check that this is a linear map. The hypothesis that $W = W_1 \oplus \ldots \oplus W_r$ says precisely that the map f is bijective, giving the required isomorphism.

Problem 2. Let V be a finite-dimensional vector space and $f, g\colon V \to V$ be linear maps. Show that fg is invertible if and only if both f and g are invertible.

Solution. If f and g are invertible, then fg is clearly invertible, as we have discussed in class. In fact $(fg)^{-1} = g^{-1}f^{-1}$: using associativity of composition of functions, we get

$$(fg)(g^{-1}f^{-1}) = f(gg^{-1})f^{-1} = f\mathrm{Id}_V f^{-1} = ff^{-1} = \mathrm{Id}_V$$

and

$$(g^{-1}f^{-1})(fg) = g^{-1}(f^{-1}f)g = g^{-1}\mathrm{Id}_V g = g^{-1}g = \mathrm{Id}_V.$$

The interesting implication is the converse. If fg is invertible, then g is injective: if $g(u) = 0$, then $fg(u) = f(0) = 0$. Hence $\mathrm{null}(g) \subseteq \mathrm{null}(fg) = \{0\}$, where the last equality follows from the fact that fg is injective. This implies that $\mathrm{null}(g) = \{0\}$, hence g is injective. Since g is an injective linear map between vector spaces of the same dimension, this implies that g is invertible by a result we proved in class.

We can now easily see that f is invertible as well: $f = (fg)g^{-1}$ and we have already shown that the composition of two invertible maps is invertible.

Problem 3. Let V and W be finite-dimensional vector spaces and let $v \in V$. Consider

$$E = \{f \in \mathcal{L}(V, W) \mid f(v) = 0\}.$$

i) Show that E is a vector subspace of $\mathcal{L}(V, W)$.
ii) If $\dim(V) = m$, $\dim(W) = n$, and $v \neq 0$, what is $\dim(E)$?

Solution. Consider the map

$$\varphi\colon \mathcal{L}(V, W) \to W, \quad \varphi(f) = f(v).$$

This is a linear map:

$$\varphi(f + g) = (f + g)(v) = f(v) + g(v) = \varphi(f) + \varphi(g)$$

and for every $a \in F$, we have

$$\varphi(af) = (af)(v) = af(v) = a\varphi(f).$$

By definition, we have $E = \mathrm{null}(\varphi)$, hence E is a linear subspace of $\mathcal{L}(V, W)$. This proves i).

Suppose now that $v \neq 0$. By a theorem proved in class, we know that

$$\dim\big(\mathcal{L}(V, W)\big) = \dim\big(\mathrm{null}(\varphi)\big) + \dim\big(\mathrm{range}(\varphi)\big) = \dim(E) + \dim\big(\mathrm{range}(\varphi)\big).$$

Let us show that φ is surjective. Since v is nonzero, there is a basis $e_1, \ldots, e_m$ of V such that $e_1 = u$. Given any $w \in W$, there is a linear map $f\colon V \to W$ such that $f(v) = w$ and $f(e_i) = 0$ for $i \geq 2$. Therefore $\varphi(f) = w$, showing that φ is surjective. We thus conclude that

$$\dim(E) = \dim\big(\mathcal{L}(V, W)\big) - \dim(W) = mn - n = n(m - 1).$$

Problem 4. Let U, V, and W be finite-dimensional vector spaces. Show that if $f \in \mathcal{L}(U, V)$ and $g \in \mathcal{L}(V, W)$, then

$$\dim\big(\mathrm{null}(gf)\big) \le \dim\big(\mathrm{null}(g)\big) + \dim\big(\mathrm{null}(f)\big).$$

Solution. Note that if $u \in \mathrm{null}(gf)$, then $f(u) \in \mathrm{null}(g)$. Consider the map

$$\alpha\colon \mathrm{null}(gf) \to \mathrm{null}(g), \quad \alpha(u) = f(u).$$

Since f is a linear map, it is clear that α is linear, too.

By the theorem proved in class, we have

$$\dim\big(\mathrm{null}(gf)\big) = \dim\big(\mathrm{null}(\alpha)\big) + \dim\big(\mathrm{range}(\alpha)\big). \tag{1}$$

By definition of α, we have $\mathrm{null}(\alpha) \subseteq \mathrm{null}(f)$, hence

$$\dim\big(\mathrm{null}(\alpha)\big) \le \dim\big(\mathrm{null}(f)\big).$$

On the other hand, $\mathrm{range}(\alpha)$ is a subspace of $\mathrm{null}(g)$, hence

$$\dim\big(\mathrm{range}(\alpha)\big) \le \dim\big(\mathrm{null}(g)\big).$$

We thus deduce from (1) that

$$\dim\big(\mathrm{null}(gf)\big) \le \dim\big(\mathrm{null}(g)\big) + \dim\big(\mathrm{null}(f)\big).$$

Problem 5. Let V and W be finite-dimensional vector spaces and $f, g \in \mathcal{L}(V, W)$. Show that $\mathrm{null}(f) \subseteq \mathrm{null}(g)$ if and only if there is $h \in \mathcal{L}(W, W)$ such that $g = hf$.

Solution. Suppose first that there is h as in the statement. If $u \in \mathrm{null}(f)$, then

$$g(u) = h\big(f(u)\big) = h(0) = 0,$$

hence $u \in \mathrm{null}(g)$. This shows that $\mathrm{null}(f) \subseteq \mathrm{null}(g)$.

Conversely, suppose that $\mathrm{null}(f) \subseteq \mathrm{null}(g)$. Choose a basis $e_1, \ldots, e_r$ of $\mathrm{range}(f)$ and complete it to a basis $e_1, \ldots, e_n$ of W. For $1 \le i \le r$, let $u_i \in V$ be such that $f(u_i) = e_i$. Let $h\colon W \to W$ be a linear map such that $h(e_i) = u_i$ for $1 \le i \le r$ and $h(e_i) \in W$ arbitrary for $i > r$. We will show that $g = hf$.

Note that by construction we have $g(u_i) = h\big(f(u_i)\big)$ for $1 \le i \le r$. Given $u \in V$, we have $f(u) \in \mathrm{range}(f)$. Since $e_1, \ldots, e_r$ span $\mathrm{range}(f)$, we can write

$$f(u) = \sum_{i=1}^{r} a_i e_i$$

for some $a_1, \ldots, a_r \in F$. We thus have

$$f(u) = \sum_{i=1}^{r} a_i f(u_i),$$

hence $u - \sum_{i=1}^{r} a_i u_i \in \mathrm{null}(f)$. The hypothesis thus implies that $u - \sum_{i=1}^{r} a_i u_i \in \mathrm{null}(g)$, hence

$$g(u) = \sum_{i=1}^{r} a_i g(u_i) = \sum_{i=1}^{r} a_i h\big(f(u_i)\big) = h\left(\sum_{i=1}^{r} a_i f(u_i)\right) = h\big(f(u)\big).$$

Since this holds for every $u \in V$, we have $g = hf$.