Math 420

Homework Set 3

This assignment is due on Monday, October 7.

The first problem concerns the *product of finitely many vector spaces*. Let $V_1, \ldots, V_r$ be vector spaces over the field F. Recall that the Cartesian product $\prod_{i=1}^r V_i$ (also written $V_1 \times \ldots \times V_r$) consists of all n-tuples $(u_1, \ldots, u_r)$, where $u_i \in V_i$ for $1 \leq i \leq r$. On $\prod_{i=1}^r V_i$ we define addition and scalar multiplication by

$$(u_1, \ldots, u_r) + (v_1, \ldots, v_r) := (u_1 + v_1, \ldots, u_r + v_r)$$
$$\lambda(u_1, \ldots, u_r) = (\lambda u_1, \ldots, \lambda u_r).$$

It is easy to check that with these operations $\prod_{i=1}^r V_i$ is a vector space; this is called the *product* of the vector spaces $V_1, \ldots, V_r$. Note that if $V_i = F$ for all i, then we recover the vector space F^r.

Problem 1. Let $V_1, \ldots, V_r$ be vector spaces over F as above and let $V = \prod_{i=1}^r V_i$.

i) Show that if
$$W_i = \{u = (u_1, \ldots, u_r) \in V \mid u_j = 0 \text{ for all } j \neq i\},$$
then W_i is a linear subspace of V and that W_i is isomorphic to V_i.

ii) Show that $V = W_1 \oplus \ldots \oplus W_r$.

iii) Show that if W is any vector space and $W_1, \ldots, W_r$ are linear subspaces such that $W = W_1 \oplus \ldots \oplus W_r$, then W is isomorphic to $\prod_{i=1}^r W_i$.

Problem 2. Let V be a finite-dimensional vector space and $f, g\colon V \to V$ be linear maps. Show that fg is invertible if and only if both f and g are invertible.

Problem 3. Let V and W be finite-dimensional vector spaces and let $v \in V$. Consider
$$E = \{f \in \mathcal{L}(V, W) \mid f(v) = 0\}.$$

i) Show that E is a vector subspace of $\mathcal{L}(V, W)$.

ii) If $\dim(V) = m$, $\dim(W) = n$, and $v \neq 0$, what is $\dim(E)$?

Problem 4. Let U, V, and W be finite-dimensional vector spaces. Show that if $f \in \mathcal{L}(U, V)$ and $g \in \mathcal{L}(V, W)$, then
$$\dim\big(\mathrm{null}(gf)\big) \leq \dim\big(\mathrm{null}(g)\big) + \dim\big(\mathrm{null}(f)\big).$$

Problem 5. Let V and W be finite-dimensional vector spaces and $f, g \in \mathcal{L}(V, W)$. Show that $\mathrm{null}(f) \subseteq \mathrm{null}(g)$ if and only if there is $h \in \mathcal{L}(W, W)$ such that $g = hf$.