Homework Set 3

Solutions are due Monday, November 9th.

Problem 1. Let $\mathcal{F}$ be a presheaf on the topological space X. Recall that $\mathcal{F}^+$ is the sheaf defined as follows: $\mathcal{F}^+(U)$ consists of all maps $s\colon U \to \sqcup_x \mathcal{F}_x$ such that

i) $s(x) \in \mathcal{F}_x$ for all $x \in U$.
ii) For all $x \in U$, there is an open neighborhood W of X, with $W \subseteq U$, and $\alpha \in \mathcal{F}(U)$ such that $s(x) = \alpha_x$ for all $x \in W$.

Show that there is a map $\mathcal{F} \to \mathcal{F}^+$ with the following universal property: for every morphism $\eta\colon \mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf, there is a unique morphism of sheaves $\phi\colon \mathcal{F}^+ \to \mathcal{G}$ such that $\phi \circ \theta = \eta$.

Problem 2. Show that for every presheaf $\mathcal{F}$, the canonical morphism $\theta\colon \mathcal{F} \to \mathcal{F}^+$ induces an isomorphism of stalks $\theta_x\colon \mathcal{F}_x \to \mathcal{F}^+_x$ for every $x \in X$.

Problem 3. Show that if $\mathcal{F}$ is a sheaf, then the canonical morphism $\theta\colon \mathcal{F} \to \mathcal{F}^+$ is an isomorphism.

Problem 4. Let X be a topological space.

i) Suppose that $\mathcal{F}$ is a presheaf on X that satisfies the first presheaf axiom, that is, for every open cover $U = \cup_i U_i$ of an open subset U of X, and for every $s, t \in \mathcal{F}(U)$ such that $s|_{U_i} = t|_{U_i}$ for all i, we have $s = t$. Show that if $\theta\colon \mathcal{F} \to \mathcal{F}^+$ is the canonical morphism, then for every open subset U of X, $\theta(U)$ is injective.
ii) Suppose that $\mathcal{F}$ is a sub-presheaf of a sheaf $\mathcal{G}$, that is, for every U we have $\mathcal{F}(U) \subseteq \mathcal{G}(U)$, the restriction maps of $\mathcal{F}$ being induced by those of $\mathcal{G}$. For every open subset U of X, let $\mathcal{F}'(U)$ be the set of those $s \in \mathcal{G}(U)$ with the property that for every $x \in X$, there is an open neighborhood W of X contained in U such that $s|_W \in \mathcal{F}(W)$. Show that there is an isomorphism $\psi\colon \mathcal{F}^+ \to \mathcal{F}'$, such that $\psi \circ \theta$ corresponds to the map induced by inclusion $\mathcal{F} \to \mathcal{F}'$.

Problem 5. Deduce from the previous problem that if $f\colon \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of abelian groups on X, then $\mathrm{Im}(f)$ is canonically isomorphic to the subsheaf $\mathcal{F}'$ of $\mathcal{G}$, where $\mathcal{F}'(U)$ consists of those $s \in \mathcal{G}(U)$ such that for all $x \in X$, there is an open neighborhood W of x, with $W \subseteq U$, and $t \in \mathcal{F}(W)$ such that $s|_W = f(t)$.

Problem 6. Let X be a topological space, Z a closed subset of X, and $U = X \smallsetminus Z$. We denote by $i\colon Z \to X$ and $j\colon U \to X$ the inclusion maps.

i) Let $\mathcal{F}$ be a sheaf of abelian groups on X. Show that the stalk $i_*(\mathcal{F})_x$ is zero if $x \notin Z$, and it is naturally isomorphic to $\mathcal{F}_x$ if $x \in Z$. For this reason, the sheaf $i_*(\mathcal{F})$ is called the *extension by zero outside* Z of $\mathcal{F}$.

ii) Suppose now that $\mathcal{G}$ is a sheaf of abelian groups on U. Let $j_!(\mathcal{G})$ be the sheaf on X associated to the presheaf $\mathcal{G}'$ such that $\mathcal{G}'(V) = \mathcal{G}(V)$ if $V \subseteq U$, and $\mathcal{G}'(V) = 0$, otherwise (with the obvious restriction maps). Show that the stalk $j_!(\mathcal{G})_x$ is naturally isomorphic to $\mathcal{G}_x$ if $x \in U$, and it is zero, otherwise. Show that $j_!(\mathcal{G})$ is the unique sheaf on X (up to isomorphism) whose restriction to U is isomorphic to $\mathcal{G}$, and whose restriction to Z is zero; $j_!(\mathcal{G})$ is called the *extension by zero outside* U of $\mathcal{G}$.

iii) Show that given a sheaf of abelian groups $\mathcal{F}$ on X, there is an exact sequence

$$0 \to j_!(\mathcal{F}|_U) \to \mathcal{F} \to i_*(\mathcal{F}|_Z) \to 0.$$